NETAJI SUBHAS OPEN UNIVERSITY

STUDY MATERIAL

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PG (MT) : IX : B (i)

Special Paper :
Pure Mathematics
Advanced Topology

PREFACE

In the curricular structure introduced by this University for students of Post-Graduate Degree Programme, the opportunity to pursue Post-Graduate course in any subject introduced by this University is equally available to all learners. Instead of being guided by any presumption about ability level, it would perhaps stand to reason if receptivity of a learner is judged in the course of the learning process. That would be entirely in keeping with the objectives of open education which does not believe in artificial differentiation.

Keeping this in view, study materials of the Post-Graduate level in different subjects are being prepared on the basis of a well laid-out syllabus. The course structure combines the best elements in the approved syllabi of Central and State Universities in respective subjects. It has been so designed as to be upgradable with the addition of new information as well as results of fresh thinking and analysis.

The accepted methodology of distance education has been followed in the preparation of these study materials. Co-operation in every form of experienced scholars is indispensable for a work of this kind. We, therefore, owe an enormous debt of gratitude to everyone whose tireless efforts went into the writing, editing and devising of proper lay-out of the materials. Practically speaking, their role amounts to an involvement in 'invisible teaching'. For, whoever makes use of these study materials would virtually derive the benefit of learning under their collective care without each being seen by the other.

The more a learner would seriously pursue these study materials, the easier it will be for him or her to reach out to larger horizons of a subject. Care has also been taken to make the language lucid and presentation attractive so that they may be rated as quality self-learning materials. If anything remains still obscure or difficult to follow, arrangements are there to come to terms with them through the counselling sessions regularly available at the network of study centres set up by the University.

Needless to add, a great deal of these efforts is still experimental–in fact, pioneering in certain areas. Naturally, there is every possibility of some lapse or deficiency here and there. However, these do admit of rectification and further improvement in due course. On the whole, therefore, these study materials are expected to evoke wider appreciation the more they receive serious attention of all concerned.

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Group
B
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Unit–I ❑ Compactness

Introduction

In this chapter we mainly deal with the notion of compactness and some of its variants. We start with the idea of nets and filters which was in the Topology course in PG–I, and present some more definitions like cluster points of nets, subnets, ultrafilters etc. Which will help us to establish some more characterizations of compactness in topological spaces. Next, the notions of three more types of compactness, namely countable compactness, Frechet compactness and sequential compactness are introduced which arise naturally from equivalent criteria of compactness in Real line with which you are already aware of. In a topological space all the four types of compactness turn out to be distinct and we establish their interrelationships.

In the remaining part of the chapter, we deal with compactness in stronger structures. First, we consider metric spaces and establish equivalent criteria of compactness by showing that all the four types of compactness are equivalent in metric spaces.

1.1 More on nets and filters

First, recall the following definitions from the earlier course on Topology.

Definition. Let $(D, \geq)$ be a directed set and X be a non-empty set. A mapping $s : D \rightarrow X$ is called a net in X. It is denoted by $\{s_n : n \in D\}$ or simply by $\{s_n\}_n$

A net $\{s_n\}_n$ is said to be eventually in $A \subset X$ if $\exists$ $n_0 \in D$ such that $s_n \in A$, $\forall n \in D$ with $n \geq n_0$.

A net $\{s_n\}_n$ is said to be frequently in $A \subset X$ if for each $m \in D$, $\exists$ *an* $n \in D$ with $n \geq m$ such that $s_n \in A$.

Definition. Let X be a topological space. A net $\{s_n\}_n$ is said to converge to $x_0 \in X$ if $\{s_n\}_n$ is eventually in every neighbourhood of x_0 and we write $\lim s_n = x_0$, x_0 is called a limit point or just a limit of $\{s_n\}_n$.

Definition. A point x_0 in a topological space X is said to be a cluster point of the net $\{s_n\}_n$ if it is frequently in every neighbourhood of x_0.

From the definition, it is clear that if a net $\{s_n\}_n$ is convergent then its limit points are the only cluster points of the net. But existence of a cluster point does not necessarily mean that the net is convergent. You have already come across such examples. Recall that taking $D = \mathbb{N}$, we had non-convergent sequences which have convergent subsequences and the limits of those convergent subsequences are in fact cluster points. This takes us to the next definition.

Definition. A net $\{t_\alpha : \alpha \in E\}$ is said to be a subnet of the net $\{s_n : n \in D\}$ if there is a mapping $i : E \to D$ such that

(a) $t = s o i$,

(b) for any $m \in D$ there is $\alpha_0 \in E$ with the property that $i(\alpha) \geq m$ for all $\alpha \in E$ with $\alpha \geq \alpha_0$.

Theorem. Let X be a topological space and $\{s_n : n \in D\}$ be a net in X. A point $x_0 \in X$ is a cluster point of $\{s_n : n \in D\}$ iff some subnet of $\{s_n\}_n$ converges to x_0.

Proof. Let x_0 be a cluster point of the net $\{s_n : n \in D\}$. Denote by N_{x_0} the family of all neighbourhoods of x_0 and let $E = \{(U, n) : n \in D \text{ and } U \in N_{x_0}\}$. For (U, n) and (V, p) in E, define $(U, n) \geq (V, p)$ iff $U \subset V$ and $n \geq p$ in $(D, \geq)$. It is easy to verify that $(E, \geq)$ is a directed set.

Let $(U, m) \in E$. Since x_0 is a cluster point of $\{s_n : n \in D\}$ it is frequently in U. So there is an element $p_{(U, m)}$ in D with $p_{(U, m)} \geq m$ such that $s_{p(U,m)} \in U$. Now define the mappings $i. E \to D$ and $t : E \to X$ as follows : $i(U, m) = p_{(U, m)}$ and $t(U, m) = s_{p(U,m)}$. Then $(s o i)(U, m) = s(i(U, m)) = s_{p(U,m)}$. so $t = s o i$. Finally let $m \in D$. Choose any $U \in Nx_0$ so that $(U, m) \in E$. Now, let $(V, n) \in E$ and $(V, n) \geq (U, m)$. Then $i(V, n) = p_{(V, n)} \geq n \geq m$. This shows that $\{t_{(U, m)} : (U, m) \in E\}$ is a subnet of the net $\{s_n : n \in D\}$.

Now let U be any neighbourhood of x_0. Choose any $m \in D$ so as to get an element $(U, m) \in E$. Now, for any $(V, n) \in E$ with $(V, n) \geq (U, m)$, we have $t_{(V, n)} = s_{p(V,n)} \in V \subset U$ which shows that the net $\{t_{(U, m)} : (U, m) \in E\}$ converges to x_0.

Next suppose that some subnet $\{t_\alpha : \alpha \in E\}$ of the net $\{s_n : n \in D\}$ converges to x_0. Then there is a mapping $i : E \to D$ satisfying the conditions for a subnet. Let U be any neighbourhood of x_0 and $m \in D$. Since $\{t_\alpha : \alpha \in E\}$ converges to x_0, $\exists\ \alpha_1 \in E$ such that $t_\alpha \in U, \forall\ \alpha \geq \alpha_1, \alpha \in E$. Again by (*b*) of the above definition $\exists\ \alpha_2 \in E$ such that $i(\alpha) \geq m\ \forall\ \alpha \geq \alpha_2, \alpha \in E$. Choose $\alpha_0 \in E$ with $\alpha_0 \geq \alpha_1, \alpha_2$. Take $\alpha \in E$ with $\alpha \geq \alpha_0$. Then $i(\alpha) \geq m$ and $t_\alpha = (\text{soi})(\alpha) = s_{i(\alpha)} \in U$. So the net $\{s_n : n \in D\}$ is frequently in U. Hence x_0 is a cluster point of $\{s_n : n \in D\}$.

Exercise A net $\{s_n : n \in D\}$ is called a maximal net (or an ultranet) in X if for any $A \subset X$ it is either eventually in A or in $X \backslash A$. Prove that if x_0 is a cluster point of a maximal net $\{s_n : n \in D\}$ then it is convergent to x_0.

Solution : Let U be any neighbourhood of the point x_0. Since $\{s_n : n \in D\}$ is maximal, so either it is eventually in U or eventually in $X \backslash U$. If possible, suppose that it is eventually in $X \backslash U$. Then $\exists\ m \in D$ such that $s_n \in X \backslash U$ for all $n \in D$, with $n \geq m$. But as x_0 is a cluster point of $\{s_n : n \in D\}$, we can find a $p \geq m$ such that $s_p \in U$ which is a contradiction. Therefore $\{s_n : n \in D\}$ is eventually in U. Since this is true for every neighbourhood U of x_0, so $\{s_n : n \in D\}$ converges to x_0.

We now move to the idea of filters. Recall the basic definitions.

Definition. A nonempty family $\mathscr{F}$ of subsets of X is called a filter in X if (i) $\phi \notin \mathscr{F}$, (ii) $A, B \in \mathscr{F} \Rightarrow A \cap B \in \mathscr{F}$, (iii) $A \in \mathscr{F}, A \subset B \Rightarrow B \in \mathscr{F}$.

A filter $\mathscr{F}$ is said to converge to x_0 in a topological space X if every neighbourhood of x_0 belongs to $\mathscr{F}$.

Definition. A point $x_0 \in X$ is called a cluster point of a filter $\mathscr{F}$ if for every neighbourhood U of x_0 and $F \in \mathscr{F}$, $U \cap F \neq \phi$ or equivalently $x_0 \in \overline{F}$, $\forall\ F \in \mathscr{F}$.

Definition. A filter $\mathscr{F}$ in X is said to be an ultrafilter if it is not properly contained in any other filter in X.

We will now prove some interesting results about ultrafilters.

Theorem : Let X be a non-empty set and $\hat{\mathscr{F}}$ be a family of subsets of X with finite intersection property. Then there exists an ultrafilter $\mathscr{F}^*$ in X containing $\hat{\mathscr{F}}$.

Proof : Let $\mathbb{C}$ denote the collection of all families of subsets of X with finite intersection property and containing the family $\hat{\mathscr{F}}$. For $\mathscr{F}_1$, $\mathscr{F}_2$ in $\mathbb{C}$, Let $\mathscr{F}_1 \geq \mathscr{F}_2$, iff $\mathscr{F}_2 \subset \mathscr{F}_1$. It is easy to see that $(\mathbb{C}, \geq)$ is a partially ordered set.

Let $\mathbb{B}$ be any totally ordered subset of $\mathbb{C}$. Write $\mathscr{F}_0 = \cup\{\mathscr{F} : \mathscr{F} \in \mathbb{B}\}$. Clearly $\hat{\mathscr{F}} \subseteq \mathscr{F}_0$. Let $\{A_1, A_2,..., A_n\}$ be any finite subfamily of $\mathscr{F}_0$. Without loss of generality, suppose $A_i \in \mathscr{F}_i$ (i = 1, 2, ..., n) where $\mathscr{F}_i \in \mathbb{B}$. Since $\mathbb{B}$ is totally ordered, $\exists$ a $p \in N$, $1 \leq p \leq n$, such that $\mathscr{F}_p \geq \mathscr{F}_i$ $\forall$ i = 1, 2, ... , n. Then $A_1, A_2,... A_n \in \mathscr{F}_p$ and so $A_1 \cap A_2 \cap .. \cap A_n \neq \phi$. Thus $\mathscr{F}_0$ also has the finite intersection property and hence $\mathscr{F}_0 \in \mathbb{C}$. Clearly $\mathscr{F}_0$ is an upper bound of $\mathbb{B}$. Therefore by Zorn's Lemma $\mathbb{C}$ has a maximal element $\mathscr{F}^*$ (say).

Clearly $\phi \notin \mathscr{F}^*$. Let $A, B \in \mathscr{F}^*$. If $\mathscr{F}_0 = \mathscr{F}^* \cup \{A \cap B\}$ then $\mathscr{F}_0$ has fip and contains $\hat{\mathscr{F}}$. So $\mathscr{F}_0 \in \mathbb{C}$. But as $\mathscr{F}^*$ is maximal, we must have $\mathscr{F}_0 = \mathscr{F}^*$ and hence $A \cap B \in \mathscr{F}^*$. Again let $A \in \mathscr{F}^*$ and $A \subset B$. By similar argument we can show that $B \in \mathscr{F}^*$. Therefore $\mathscr{F}^*$ is a filter.

Finally, note that if $\mathscr{F}'$ is any filter containing $\mathscr{F}^*$ then $\hat{\mathscr{F}} \subset \mathscr{F}^* \subset \mathscr{F}'$ and so $\mathscr{F}' \in \mathbb{C}$. Since $\mathscr{F}^*$ is a maximal element of $\mathbb{C}$ so we must have $\mathscr{F}^* = \mathscr{F}'$. This proves that $\mathscr{F}^*$ is an ultrafilter.

Theorem : A filter $\mathscr{F}^*$ is an ultrafilter in X iff any subset A of X which intersects every member of $\mathscr{F}^*$ belongs to $\mathscr{F}^*$.

Proof : First suppose that $\mathscr{F}^*$ is an ultrafilter in X. Let A be a subset of X which intersects every member of $\mathscr{F}^*$.

Let $\mathscr{F}_0 = \{C \subset X : A \cap B \subset C$ for some $B \in \mathscr{F}^*\}$. Clearly $\phi \notin \mathscr{F}_0$, $\mathscr{F}^* \subset \mathscr{F}_0$ and $A \in \mathscr{F}_0$. Let $C_1, C_2 \in \mathscr{F}_0$. Then $A \cap B_1 \subset C_1$ and $A \cap B_2 \subset C_2$ for $B_1, B_2 \in \mathscr{F}^*$. Then $B = B_1 \cap B_2 \in \mathscr{F}^*$ and we have $C_1 \cap C_2 \supset (A \cap B_1) \cap (A \cap B_2) = A \cap (B_1 \cap B_2) = A \cap B$ which implies $C_1 \cap C_2 \in \mathscr{F}_0$. Again if $C \in \mathscr{F}_0$ and $C \subset C'$ $(\subset X)$ then $\exists$ $B \in \mathscr{F}^*$ such that $A \cap B \subset C$ and so $A \cap B \subset C'$ which implies $C' \in \mathscr{F}_0$. Therefore $\mathscr{F}_0$ is a filter in X. Since $\mathscr{F}^*$ is an ultrafilter so $\mathscr{F}^* = \mathscr{F}_0$ and so $A \in \mathscr{F}^*$.

Next suppose that the given condition holds. Let $\mathscr{F}$ be any filter in X containing $\mathscr{F}^*$. Let $A \in \mathscr{F}$. If $B \in \mathscr{F}^*$ then $B \in \mathscr{F}$ and so $A \cap B \neq \phi$. So by our hypothesis $A \in \mathscr{F}^*$. This shows that $\mathscr{F} = \mathscr{F}^*$. Hence $\mathscr{F}^*$ is an ultrafilter.

Exercise : Let $\mathscr{F}^*$ be an ultrafilter in X and A, B be two subsets of X such that $A \cup B \in \mathscr{F}^*$. Then either $A \in \mathscr{F}^*$ or $B \in \mathscr{F}^*$.

Solution : Suppose that $A \notin \mathscr{F}^*$. Consider the family $\mathscr{F}_0 = \{C \subset X : A \cup C \in \mathscr{F}^*\}$. Then $B \in \mathscr{F}_0$. Since $A \notin \mathscr{F}^*$, $\phi \notin \mathscr{F}_0$. Let $C_1, C_2 \in \mathscr{F}_0$. Then $A \cup C_1, A \cup C_2 \in \mathscr{F}^*$ and so

$A \cup (C_1 \cap C_2) = (A \cup C_1) \cap (A \cup C_2) \in \mathscr{F}^*$. This proves that $C_1 \cap C_2 \in \mathscr{F}_0$. Again let $C \in \mathscr{F}_0$ and $C \subset C'$. Then $A \cup C \in \mathscr{F}^*$. But since $A \cup C \subset A \cup C'$, so $A \cup C' \in \mathscr{F}^*$ which then implies $C' \in \mathscr{F}_0$. So $\mathscr{F}_0$ is a filter in X.

Finally as we can see, $C \in \mathscr{F}^* \Rightarrow A \cup C \in \mathscr{F}^*$ and so $C \in \mathscr{F}_0$. Thus $\mathscr{F}^* \subset \mathscr{F}_0$. But as $\mathscr{F}^*$ is an ultrafilter so $\mathscr{F}_0 = \mathscr{F}^*$. Therefore $B \in \mathscr{F}^*$.

Exercise : A filter $\mathscr{F}^*$ in X is an ultrafilter iff for any $A \subset X$ either $A \in \mathscr{F}^*$ or $X \setminus A \in \mathscr{F}^*$.

Solution. First suppose that $\mathscr{F}^*$ is an ultrafilter. Let $A \subset X$. Since $X \in \mathscr{F}^*$ and $X = A \cup (X \setminus A)$ so either $A \in \mathscr{F}^*$ or $X \setminus A \in \mathscr{F}^*$. Conversely, suppose that the given condition holds. Let $\mathscr{F}$ be a filter containing $\mathscr{F}^*$. If $\mathscr{F}^* \subseteq \mathscr{F}$ then we can choose some $A \in \mathscr{F}$ such that $A \notin \mathscr{F}^*$. But then by the given condition $X \setminus A \in \mathscr{F}^*$ which implies $X \setminus A \in \mathscr{F}$. Then $\phi = A \cap (X \setminus A) \in \mathscr{F}$ which is a contradiction. Hence $\mathscr{F}^* = \mathscr{F}$ and so $\mathscr{F}^*$ must be an ultrafilter.

1.2 Compactness

We first recall the following definitions and a result from earlier Topology course.

Definition : A topological space (X, τ) is said to be compact if every open covering of X has a finite subcovering.

Compactness can be characterised in terms of "the finite intersection property" of closed sets.

Definition : (Finite intersection property) : A collection of subsets $\{F_v : v \in \wedge\}$ of a given set X ($\wedge$ being an indexing set) is said to possess the finite intersection property, if every finite sub-collection of $\{F_v\}$ has non-empty intersection.

Theorem : A topological space (X, τ) is compact if and only if for every collection of closed sets $\{F_\nu : \nu \in \wedge\}$ in (X, τ), possessing the finite intersection property, the intersection $\cap\{F_\nu : \nu \in \wedge\}$ of the entire collection is non-empty.

We now prove the following characterizations of compactness.

Theorem : Let (X, τ) be a topological space. Then the following statements are equivalent.

(i) X is compact

(ii) Every filter in X has a cluster point

(iii) Every ultrafilter in X converges.

Proof : (i) $\Rightarrow$ (ii) : Suppose that X is compact. Let $\mathscr{F}$ be any filter in X. Let $\mathscr{F}^* = \{\overline{A} : A \in \mathscr{F}\}$. Then $\mathscr{F}^*$ is a family of closed sets with finite intersection property. Since X is compact, so

$$\cap\{\overline{A} : A \in \mathscr{F}\} \neq \phi.$$

Choose a point x_0 in $\cap\{\overline{A} : A \in \mathscr{F}\}$. Then $x_0 \in \overline{A}, \forall A \in \mathscr{F}$ and from definition x_0 is a cluster point of $\mathscr{F}$.

(ii) $\Rightarrow$ (iii) : Let $\hat{\mathscr{F}}$ be an ultrafilter in X. By (ii), $\hat{\mathscr{F}}$ has a cluster point x_0 in X. Let U be any neighbourhood of x_0. Then $U \cap F \neq \phi\ \forall F \in \hat{\mathscr{F}}$. But then we must have $U \in \hat{\mathscr{F}}$. This shows that $\hat{\mathscr{F}}$ converges to x_0.

(iii) $\Rightarrow$ (i) : Finally suppose that (iii) holds. Let $\mathscr{F}$ be a family of closed sets in X with finite intersection property. Then there exixts an ultrafilter $\hat{\mathscr{F}}$ containing $\mathscr{F}$. By (iii), $\hat{\mathscr{F}}$ converges to a point $x_0 \in X$. Then for any neighbourhood U of x_0, $U \in \hat{\mathscr{F}}$. Take any $F \in \mathscr{F}$. Then $F \in \hat{\mathscr{F}}$ and so $U \cap F \neq \phi$. This shows that x_0 is a limit point of F and so $x_0 \in \overline{F}$. But since each $F \in \mathscr{F}$ is closed, $x_0 \in \overline{F} = F$. This is true for any $F \in \mathscr{F}$ and so

$$\cap\{F : F \in \mathscr{F}\} \neq \phi.$$

This proves that X is compact.

We now use the concept of ultrafilter, developed so far, to prove the following important theorem due to Tychonoff.

Theorem (Tychonoff Product Theorem).

Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of topological spaces. Then the topological product space X is compact iff each X_α is so.

Proof : If X is compact, then clearly each factor space X_α, being continuous image of X under the projection map $p_\alpha : X \to X_\alpha$, is compact.

Conversely, let each space X_α be compact. By the above theorem it suffices to show that any ultrafilter $\mathscr{F}$ on X converges in X. For each $\alpha \in \Lambda$, $\mathscr{B}_\alpha = \{p_\alpha(F) : F \in \mathscr{F}\}$ is clearly a base for a filter $\mathscr{F}_\alpha$ on X_α. We claim that $\mathscr{F}_\alpha$ is an ultrafilter on X_α. For this, we need to show that for any subset A of X_α, either $A \in \mathscr{F}_\alpha$ or $X_\alpha \setminus A \in \mathscr{F}_\alpha$. Let us write $B = p_\alpha^{-1}(A)$. Since $\mathscr{F}$ is an ultrafilter on X, either $B \in \mathscr{F}$ or $X \setminus B \in \mathscr{F}$. Consequently, either $A = p_\alpha(B) \in \mathscr{B}_\alpha \subseteq \mathscr{F}_\alpha$ or $(X \setminus A) \in \mathscr{F}_\alpha$. Hence $\mathscr{F}_\alpha$ is an ultrafilter in X_α, for each $\alpha \in \Lambda$. As each X_α is compact, $\mathscr{F}_\alpha$ converges to some $x_\alpha \in X_\alpha$, for each $\alpha \in \Lambda$. Then $\mathscr{F}$ converges to the point $x = (x_\alpha)_{\alpha \in \Lambda}$ in X and hence X is compact.

1.3 Countable Compactness

We now look into another type of compactness which is weaker than compactness but is equivalent to compactness in the real line.

Definition : A topological space (X, τ) is said to be countably compact, if every countable open covering of X has a finite subcovering.

We shall obtain several necessary and sufficient conditions for a topological space to be countably compact. One such condition is given in terms of the concept of cluster point of a sequence. A point p is called a cluster point of an infinite sequence $\{x_n : n = 1, 2, ...\}$ in a topological space (X, τ) if, for any given open set U, containing p, and any positive integer r, there always exists a positive integer $m > r$, such that $x_m \in U$.

Theorem : For a topological space (X, τ) the following conditions are equivalent:

(a) (X, τ) is countably compact.

(b) Every countable aggregate of closed sets, possessing the finite intersection property, has a non-empty intersection in (X, τ).

(c) Every descending chain of non-empty closed sets, $F_1 \supset F_2 \supset ...$, has a non-empty intersection in (X, τ). (Cantor's intersection theorem)

(d) Every infinite sequence in X has a cluster point in X.

(e) Every infinite set $S \subset X$ has an w-accumulation point in X.

Proof : (a) ⇒ (b) : This is quite similar to the corresponding theorem on compactness.

(b) ⇒ (c) : Clearly $\{F_n : n \in \mathbb{N}\}$ is a countable collection of closed sets with the finite intersection property and hence by (b), $\bigcap_{n=1}^{\infty} F_n \neq \phi$

(c) ⇒ (d) : Let $\{x_n\}$ be a sequence in X and Let $A_n = \{x_m : m > n\}$ for each $n \in \mathbb{N}$. Clearly $\{\overline{A}_n : n \in \mathbb{N}\}$ is a descending sequence of nonempty closed sets in X. By (c), there is a point $a \in \bigcap_{n=1}^{\infty} \overline{A}_n$. We claim that a is a cluster point of the given sequence. Indeed, for any open neighbourhood U of a and any $m \in \mathbb{N}$, we have some $x_n \in A_m \cap U$ as $a \in \overline{A}_m$. Then $n > m$ such that $x_n \in U$.

(d) ⇒ (a). If possible, suppose (a) does not hold. Then there is a countable open covering $\{U_n : n \in \mathbb{N}\}$ of X having no finite subcovering. Let $C_n = X \backslash (U_1 \cup ... \cup U_n)$. Clearly, $\{C_n : n \in \mathbb{N}\}$ is a descending sequence of nonempty closed sets in X. Choose $x_n \in C_n$, for each $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ has a cluster point x (say) in X (by (d)). Since $\{U_n : n \in \mathbb{N}\}$ is a cover of X, $\exists\ m \in \mathbb{N}$ such that $x \in U_m$. Now, $n > m \Rightarrow C_n \subseteq C_m \Rightarrow x_n \in C_m \Rightarrow x_n \notin U_m$. Thus x cannot be a cluster point of $\{x_n : n \in \mathbb{N}\}$, a contradiction.

To complete the proof, it now suffices to prove '(d) ⇔ (e)' which we do as follows:

(d) ⇒ (e) : Given an infinite set S in X, we can always construct a sequence $\{a_n\}$ in S such that $a_n \neq a_m$ for $n \neq m$ $(n, m \in \mathbb{N})$. By (d), this sequence has a cluster point p (say) in X. Then erery neighbourhood of ρ contains infinitely many terms of the sequence, i.e., contains infinitely many points of S. Hence p is an w-accumulation point of S.

(e) ⇒ (d) : Let $\{a_n\}$ be a sequence in X and let A be the set formed by the values taken by the sequence. If A is a finite set, then there is an element a such that $a_n = a$ for infinitely many natural numbers n. Obviously, then a is a cluster point of the sequence. If A is an infinite set, then by (e), A has an w-accumulation point p (say). Obviously, p is then a cluster point of $\{a_n\}$.

Exercise

(a) A subspace of a countably compact space need not be countably compact.

(b) Every closed subspace of a countably compact space is countably compact.

(c) The union of a finite collection of countably compact subspaces of a topological space is a countably compact subspace.

Solution : (a) The closed unit interval [0, 1] is compact, by Heine-Borel theorem, hence it is also countably compact. The subspace (0, 1) of [0, 1], is, however, not countably compact.

1.4 Sequentially Compact and Frechet Compact spaces

Finally we look into two types of compactness, one of which is defined by using sequences and the other defined by using the idea of limit points of sets.

Definition : (Sequentially Compact) : A topological space (X, τ) is said to be sequentially compact, if every infinite sequence in X contains a convergent subsequence.

Definition : (Frechet Compact) : A topological space (X, τ) is said to be Frechet compact (or *B-W* compact i.e., Bolzano-Weierstrass compact), if every infinite subset of X has an accumulation point.

Theorem : (a) Every closed subspace of a sequentially compact space is sequentially compact.

(b) Every closed subspace of a Frechet compact space is Frechet compact.

It follows from the following example that :

(i) a subspace of a sequentially compact space need not be sequentially compact, and (ii) a subspace of a Frechet compact space need not be Frechet compact.

Example : Let $\mathbb{R}$ be the set of reals, and v consists of (i) all those subsets of $\mathbb{R}$, which do not contain 0, and (ii) the 4 subsets $\mathbb{R}\setminus\{1, 2\}$, $\mathbb{R}\setminus\{1\}$, $\mathbb{R}\setminus\{2\}$, and $\mathbb{R}$. Then $(\mathbb{R}, \nu)$ is a first countable, Lindeloff space. Any open covering V of $\mathbb{R}$ must include at least one of the sets in (ii) (in order that 0 may be covered). Let G be such a set for the open covering V of $\mathbb{R}$, then $X\setminus G$ consists of at most two points 1 & 2. Let H_1 & H_2 be two members of V, containing the points 1 & 2 respectively. Then $\{G, H_1, H_2\}$ forms a finite subcovering of V for $\mathbb{R}$. Hence $(\mathbb{R}, \nu)$ is compact. let $S = \mathbb{R}\setminus\{0\}$, then the subspace $(S, \nu_s\}$ is not a Lindeloff space.

As ν_s is the discrete topology on S, S is an infinite set having no accumulation point in S. Hence the subspace (S, ν_s) is not Frechet Compact. The space $(\mathbb{R}, \nu)$ is also sequentially compact. In fact, any infinite sequence $\{x_i ; i = 1, 2 ...\}$ in $\mathbb{R}$ is of any one of the following two types :

(i) $x_i \neq 1$ and 2 for all i, except for finitely many values of i and the sequence $\{x_i : i = 1, 2 ...\}$ is itself convergent. Converging to the limit 0;

(ii) $x_i = 1$ or 2 for infinitely many values of i, and then there exists an infinite subsequence of $\{x_i : i = 1, 2 ...\}$, which converges to the limit 1 or 2.

1.5 Mutual dependence of different types of compactness

Now we investigate the interrelationships between the four types of compactness we have come across.

Theorem : (a) Every compact space is countably compact and also a Lindeloff space.

(b) A countably compact Lindeloff space is compact.

Proof : (a) Let (X, τ) be a compact space. Since for every open covering of X, there exists a finite sub-covering, the same is true for every countable open covering. Hence (X, τ) is countably compact. Also, since a finite sub-covering is necessarily a countable sub-covering, it follows that (X, τ) is also a Lindeloff space.

(b) Let (X, τ) be a countably compact, Lindeloff space. Let U be any open covering of X. As (X, τ) is a Lindeloff space, there exists a countable subcovering V of U for X. Again, since (X, τ) is countably compact, for the countable open covering V of X, there exists a finite subcovering W. Then W is a finite sub-covering of U for X. Hence the space (X, τ) is compact.

Theorem : (a) A countably compact space is Frechet compact.

(b) Any Frechet compact T_1-space is countably compact.

Proof : (a) Let (X, τ) be a countably compact space. Then every infinite subset S of X has an w-accumulation point in X ; thus S has an accumulation point in X. Hence (X, τ) is a Frechet compact space.

(b) Let (X, τ) be a Frechet compact, T_1-space, and S be an infinite subset of X. As (X, τ) is Frechet compact, S has an accumulation point x (say) in X and since (X, τ) is a T_1–space, the accumulation point x is an ω-accumulation point. Hence (X, τ) is countably compact.

Theorem : (a) A sequentially compact space is countably compact.

(b) Any countably compact, first countable space is sequentially compact.

Proof : (a) Let (X, τ) be a sequentially compact space and let $\{x_i : i = 1, 2, ...\}$ be any infinite sequence in X. Then the sequence $\{x_i : i = 1, 2 ...\}$ contains a convergent subsequence. The limit of the convergent subsequence is a cluster point of the sequence $\{x_i : i = 1, 2, ...\}$. Hence (X, τ) is countably compact.

(b) Let (X, τ) be a countably compact, first countable space. Let $\{x_i : i = 1, 2, ...\}$ be an infinite sequence in X. Since (X, τ) is countably compact, it is also Frechet compact; hence the infinite sequence $\{x_i : i = 1, 2 ...\}$ has an accumulation point x (say) in X. Again, since the space (X, τ) is first countable, it follows that there exists a sub-sequence $\left\{x_{k_i} : i = 1,2,...\right\}$ of the sequence $\{x_i : i = 1, 2, ...\}$, such that lim $x_{k_i} = x$. Thus the sequence $\{x_i ; i = 1, 2, ...\}$ contains a convergent subsequence $\left\{x_{k_i} : i = 1,2,...\right\}$. Hence the space (X, τ) is sequentially compact.

Note : In proving the part (b) of the above theorem, we have merely used the property that (X, τ) is Frechet compact (in place of its countable compactness). Hence, every Frechet compact, first countable space is sequentially compact.

In view of the fact that a second countable space is first countable and also a Lindeloff space, it follows that :

Theorem : For a second countable T_1-space, any one of the four properties (i) compactness, (ii) countable compactness, (iii) sequential compactness and (iv) Frechet compactness, implies the other three.

If can be shown by constructing suitable counter-examples that no other direct implication exists between the Lindeloff property and the four compactness properties.

Exercise : Give an example of a second countable (and hence Lindeloff), Frechet compact space that is not countably compact.

Solution : Consider the topological space $(\mathbb{N}, \tau)$. Here $\mathbb{N}$ is the set of natural numbers and τ is the odd-even topology on $\mathbb{N}$. The topology τ is generated by the base $B = \{\phi\} \cup \{(2n - 1, 2n) : n = 1, 2, ...\}$. The space $(\mathbb{N}, \tau)$ is second countable, since the open base B of τ is countable. Also B forms a countable open covering of $\mathbb{N}$, for which there is no finite sub covering, hence $(\mathbb{N}, \tau)$ is not countably compact.

Let P be an infinite subset of $\mathbb{N}$ and let $p \in P$. Let now, $x = p + 1$ if p is odd, and $x = p - 1$ if p is even. Then every open set, containing x, also contains p ; hence x is an accumulation point of P in $\mathbb{N}$. Consequently, the space $(\mathbb{N}, \tau)$ is Frechet compact.

Exercise : Give an example of a compact Hausdorff space that is not sequentially compact.

Solution : Let I denote the closed unit interval [0, 1] with the subspace topology σ_1, induced by the usual topology σ of the real number space $(\mathbb{R}, \sigma)$. Let $(X, \tau) = \Pi \{I_r : I_r = I, r\in\mathbb{R}\}$. Thus, $X = I^I$ is the uncountable product of I. Hence X is compact and T_2, since I is so. Again X is not sequentially compact, since the sequence of functions $f_n \in X$ defined by $f_n(x)$ = the nth digit in the binary expansion of x, has no convergent sub-sequence.

For, suppose $\left\{f_{n_k}\right\}$ is a subsequence which converges to a point $f \in X$. Then, for each $x\in I$, $f_{n_k}(x)$ converges in I to $f(x)$. Let $p \in$ I have the property that $\alpha_{n_k}(p) = 0$ or 1 according as whether k is odd or even. Then $\{\alpha_{n_k}(p)\}$ is 0, 1, 0, 1, ... which cannot converge.

1.6 Compactness in Metric spaces :

In the final section of the chapter we consider the four types of compactness in metric spaces.

Theorem : In a metric space (X, d), the concepts of second countability, separability and Lindeloffness are equivalent.

Proof : Suppose (X, d) is separable. Let $A = \{x_i\}$ be a countable dense subset of X. Let $B = \{B(x_i, r)$; r-rational, $i = 1, 2, ...\}$. Then B is countable. Also if U is any open set and $x \in U$, $\exists$ an $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$. Since A in dense in X, we can choose x_k s.t. $d(x_k, x) < \varepsilon/2$. Then it is easy to show that $x \in B(x_k, \varepsilon/2) \subset U$ and so B is also a base in X. Hence X is second countable.

It is known that a second countable space is Lindeloff. Now let (X, d) be Lindeloff. Let $\varepsilon = \frac{1}{n}$. From the open cover $\left\{B\left(x, \frac{1}{n}\right); x \in X\right\}$ of X, we can find a countable covering $\left\{B\left(x_i, \frac{1}{n}\right): i \in \wedge\right\}$. Let $A_{\frac{1}{n}} = \{x_i ; i \in \mathbb{N}\}$. Then $A = \bigcup_{n=1}^{\infty} A_{\frac{1}{n}}$ is a countable dense subset of X and so X is separable.

We now introduce the following definition.

Definition. A finite subset F of a metric space X is called an ε-net for X if $X \subset \cup\{B(x, \varepsilon) : x \in F\}$ X is called totally bounded if it has an ε-net for every $\varepsilon > 0$.

Theorem : A countably compact metric space is totally bounded.

Proof : If possible let (X, d) be not totally bounded. Then $\exists$ an $\varepsilon > 0$ such that there is no ε-net for X. Let $p_1 \in X$. Clearly $X \not\subset B(p_1, \varepsilon)$. Choose $p_2 \in X \setminus B(p_1, \varepsilon)$. Since $\{p_1, p_2\}$ is not an ε-net, we can find $p_3 \in X$ such that $d(p_1, p_3) \geq \varepsilon$, $d(p_2, p_3) \geq \varepsilon$. Proceeding in this way we get a sequence $\{p_n\}$ of distinct points in X such that $d(p_i, p_j) \geq \varepsilon$, $i \neq j$. Since a metric space is first countable, so countable compactness of X implies sequential compactness of X. But evidently the sequence $\{p_n\}$ has no convergent subsequence in X. This contradicts the fact that X is countably compact.

Exercise : Prove that a totally bounded metric space X is separable.

Solution : For each $n \in \mathbb{N}$, X has an $\frac{1}{n}$-net F_n. Let $F = \bigcup_{n=1}^{\infty} F_n$. Then F is clearly countable. Let $x \in X$ and let $\delta > 0$ be given. Choose $m \in \mathbb{N}$ such that $\frac{1}{m} < \delta$. Now since F_m is a $\frac{1}{m}$-net, $\exists$ *a* $w \in F_m$ such that $d(x, w) < \frac{1}{m} < \delta$ i.e. $w \in B(x, \delta)$. This proves that F is also dense in X and so X is separable.

Theorem. For a metric space, compactness, countable compactness, Frechet compactness and sequential compactness are all equivalent.

Note. A totally bounded metric space is not necessarily compact. If $A = \{x \in Q : 0 \leq x \leq 1\}$ and $d^* = d_{A \times A}$ where d is the usual metric, then (A, d^*) is totally bounded but not compact.

Exercise : A set $A \subset (X, d)$ is called relatively compact if $\overline{A}$ is compact in X. Show that a relatively compact set A in a metric space (X, d) is totally bounded.

Recall that in a metric space (X, d), a sequence $\{x_n\}$ is called a Cauchy sequence if for any $\varepsilon > 0$ $\exists$ a $k \in \mathbb{N}$ s.t. $m, n \geq k \Rightarrow d(x_m, x_n) < \varepsilon$. Every convergent sequence is Cauchy. Also a Cauchy sequence having a convergent subsequence is also convergent.

Definition : (X, d) is said to be complete if every Cauchy sequence in X converges in X.

Theorem : A compact metric space is complete.

Theorem : If (X, d) is complete and totally bounded then it is compact.

Proof : We show that (X, d) is sequentially compact. Let $\{x_n\}$ be a sequence in X. Since X is totally bounded, X is contained in the union of a finite number of open balls of radius 1. At least one of them must contains a sub-sequence of $\{x_n\}$, say $\{x_{11}, x_{12}, ...\}$. Again from the property of total boundedness of X, we can find an open ball of radius $\frac{1}{2}$ which contains a subsequence of $\{x_{1n} ; n \in \mathbb{N}\}$, say $(x_{2n}, n = 1, 2, ...\}$. Proceeding in this way, by induction we obtain sequences $\{x_{ki} ; i = 1, 2, ...\}$ $(k = 1, 2...)$ each sequence is a subsequence of the predecessor and the k^{th} sequence is contained in a ball of radius $\frac{1}{k}$. It is easy to see that $\{x_{kk}; k = 1, 2, ...\}$ is a subsequence of $\{x_n\}$ which is Cauchy in X and so is convergent in X, since X is complete. Hence (X, d) is compact.

Note. The space l_2 is complete but not totally bounded and so is not compact.

Theorem : A metric space is compact iff it is complete and totally bounded.

Exercise : Prove that Lindeloff'ness is not a hereditary property.

Solution : Let X be an uncountable set and let $x_0 \in X$ be chosen. Define a topology τ on X as follows : (i) $\phi, X \in \tau$. (ii) $A\ (\subset X) \in \tau$ iff $x_0 \notin A$. First we will show that (X, τ) is a Lindeloff space. Let v be an open cover of X. Since the only open set containing x_0 is X itself, so $X \in v$ and $\{X\}$ is the required subcover. Now if $Y = X \setminus \{x_0\}$ then (Y, τ_y) is a discrete topological space. Taking v_l as the collectioin of all singletons from τ_Y, we see that v_1 cannot have a countable family which also covers Y.

Group–A
(Short questions)

1. Show that the collection $\mathscr{F}(p)$ of all subsets of a set X which contain a given element $p \in X$ is an ultrafilter on X.
2. Show that in a discrete topological space every neighbourhood filter is an ultrafilter.
3. Prove that the net associated with an ultrafilter is a maximal net.
4. Show that a filter $\mathscr{F}$ converges to a point $x \in X$ iff every ultrafilter containing $\mathscr{F}$ converges to x.
5. Give an example of a totally bounded space which is not compact.
6. Prove that continuous image of a sequentially compact set is sequentially compact.

7. Examine whether $\{\{a\}, \{b\}, \{a, b\}, X\}$ where $X = \{a, b, c\}$ is a filter.

8. Give an example to show that sequential compactness is not a hereditary property.

9. Prove that a finite union of compact subspaces of a topological space is compact.

Group–B
(Long questions)

1. Prove that every filter $\mathscr{F}$ on X is the intersection of all the ultrafilters finer than $\mathscr{F}$.

2. For an ultrafilter $\mathscr{F}$ on a set X prove that $\cap \{F : F \in \mathscr{F}\}$ is either empty or a singleton subset of X.

3. For a topological space (X, τ), prove that following are equivalent.

   (i) X is compact.

   (ii) Every net in X has a convergent subnet.

   (iii) Every maximal net in X converges in X.

4. Let (X, τ) be a topological space and $A \subset X$. Prove that A is τ-open iff A belongs to every filter which converges to a point of A.

5. If $\{f_n\}_n$ is a sequence of real valued continuous functions on a compact topological space X and $f_n \to f$ on X then prove that $f_n \to f$ uniformly on X.

6. Show that a subspace of R^n is bounded iff it is totally bounded.

7. If (X, d) is a complete metric space and $A \subset X$ is totally bounded then prove that A is relatively compact (i.e. $\overline{A}$ is compact).

8. Prove that a subnet of a subnet of a net $\{x_n : n \in D\}$ is a subnet of $\{x_n : n \in D\}$.

9. Give an example to show that the continuous image of a Frechet compact space need not be Frechet compact. If $f : X \to Y$ is a continuous bijection and X is Frechet compact, is it true that Y is also Frechet compact? Justify.

Unit–II ❑ Compactification

Introduction

In this chapter we start with the idea of local compactness. Though the definition was mentioned in the basic Topology course, here, it is dealt with in full detail as locally compact spaces are more common than compact spaces and they have many interesting properties. We then consider the notion of compactification. Though one-point compactification has been already included in the earlier course, it is again described in detail for the sake of completeness. We then consider two more ideas of compactification, which are much deeper and which have played important role in the advancement of the subject. The first type of compactification, that is dealt with, is Stone-Cech compactification which happens to be the largest Hausdorff compactification among all possible Hausdorff compactifications of a given Tychonoff space and there-in lies its importance. Another very strong result is the Stone-Cech theorem showing that any continuous function on a Tychonoff space can be extended to its Stone-Cech compactification. Finally we consider Wallman's compactification which is different from the other two compactifications in view of the use of ultrafilters in its construction.

2.1 Locally Compact spaces

A topological space is said to be locally compact if each point of the space has at least one compact neighbourhood.

Clearly every compact space is locally compact but the converse is not true, as R with usual or discrete topology is locally compact but not compact.

Theorem : 1. Let X be a locally compact space. The family of all closed compact neighbourhoods of each point x of X forms a neighbourhoods basis at x if in addition X is regular or Hausdorff.

Proof : Let $x \in X$. Denote by $\mathscr{V}$ the family of all closed and compact neighbourhoods of x. Since X is locally compact, there is a compact neighbourhood C of x.

(I) First suppose that X is a regular space. Let U be any neighbourhood of x. Then $(U \cap C)^\circ$ is an open neighbourhood of x. So there is an open neighbourhood V of x suct that $x \in V \subset \bar{V} \subset (U \cap C)^\circ$. Since $\bar{V} \subset C$ and C is compact, so $\bar{V}$ is also compact. Clearly $\bar{V}$ is then a closed compact neighbourhood of x and so $\bar{V} \in \mathscr{V}$. Also $\bar{V} \subset U$ and hence $\mathscr{V}$ is a neighbourhood basis at x.

(II) Next let X be T_2 and U be any neighbourhood of x. Let us take $W = (U \cap C)^\circ$. Then W is an open neighbourhood of x. Since C is compact and X is T_2, C is closed. Then $W \subset C \Rightarrow \bar{W} \subset C$ which implies $\bar{W}$ is compact.

Write $F = \bar{W} \setminus W$. Then F is a closed compact set and $x \notin F$. Since X is T_2, $\exists$ two open sets V_1, G_1 such that

$$x \in V_1,\ F \subset G_1 \text{ and } V_1 \cap G_1 = \phi.$$

Let $V = W \cap V_1$ and $G = \bar{W} \cap G_1$. Then V is an open neighbourhood of x, $F \subset G$ and $V \cap G = \phi$. Now

$$V \subset W \setminus G \subset \bar{W} \setminus G \subset \bar{W} \setminus F = W \subset U$$

and $\bar{V} \subset \bar{W} \setminus G \subset W \subset \bar{W} \subset C$

$= (\bar{W} \cap X \setminus G_1)$ ($\because$ $V \subset V_1 \subset X \setminus G_1$ closed), which implies that $\bar{V}$ is compact. Hence $\bar{V} \in \mathscr{V}$. Since $\bar{V} \subset U$, hence $\mathscr{V}$ forms a neighbourhood basis at x.

Exericise : Every locally compact T_2 space is regular.

Solution : Let X be a locally compact T_2 space.

Let $x \in X$ and U be a neighbourhood of x. Then proceeding exactly in the same as in the last part of the proof of theorem 1, we get two open neighbourhood V and W, and an open set G_1 such that.

$\bar{V} \subset \bar{W} \setminus G \subset W \subset U$, where $G = \bar{W} \cap G_1$. Thus $x \in V \subset \bar{V} \subset U$ which proves that X is regular.

Theorem 3 : A locally compact regular space is completely regular.

Proof : We prove the theorem by the following steps.

(I) Let A be a compact subset of X and U be an open set with $A \subset U$. Let $x \in A$. Since X is locally compact, there is a closed compact neighbourhood W_x of x (which is also closed by from theorem 1) such that $W_x \subset U$. Now the collection $\{W_x^\circ ; x \in A\}$ from an open cover A. Since A is compact, there exists a finite number of points x_1, x_2, .., x_n such that $A \subset W_{x_1}^\circ \cup W_{x_2}^\circ \cup .. \cup W_{x_n}^\circ$. Write $V = \bigcup_{i=1}^{n} W_{x_i}^\circ$. Then V is an open set containing A. Also $\overline{V} = \bigcup_{i=1}^{n} Wx_i$. So $\overline{V}$ is compact and $A \subset V \subset \overline{V} \subset U$.

(II) Let F be any closed subset of X and let $x_0 \in X \setminus F$. Since $X \setminus F$ is open, there is a closed compact neighbourhood A of x_0 with $A \subset X \setminus F$. By step (I), there is an open set V such that $\overline{V}$ is compact and

$$A \subset V \subset \overline{V} \subset X \setminus F.$$

Write $B = \overline{V} \setminus V$. Then B is also a closed compact set with $A \cap B = \phi$.

Clearly $\overline{V}$ with the relative topology, is a compact Hausdorff space and so is normal. Hence there is a continuous mapping $g : \overline{V} \to [0, 1]$

such that $g(x) = 0 \quad \forall x \in A$

$= 1 \quad \forall x \in B.$

We now define the mapping $f : X \to [0, 1]$ by

$f(x) = g(x) \quad \forall x \in \overline{V}$

$= 1 \quad \forall x \in X \setminus \overline{V}.$

We now show that f is continuous.

(a) Let $z \in V$. Choose any $\in > 0$. Since g is continuous, there is an open neighbourhood U of z in the space $\overline{V}$ such that $|g(x) - g(z)| < \in$, $\forall x \in U$. we can write $U = \overline{V} \cap G$, where G is an open set in X. Let $W = V \cap G$. Then W is an open neighbourhood of z in X and we have

$|f(x) - f(z)| = |g(x) - g(z)| < \in$, $\forall x \in W$ (since $W \subset U$) and so f is continuous at z.

(b) Let $z \in X \setminus \bar{V} = W$ (say). Let $\in > 0$ be given. Clearly W is an open neighbourhood of z in X and we have

$|f(x) - f(z)| = |1 - 1| = 0 < \in, \ \forall x \in W.$

So f is continuous at z.

(c) Finally let $z \in \bar{V} \setminus V$. Then $f(z) = g(z) = 1$. Let $\in > 0$ be given. Since g is continuous, there is an open neighbourhood U of z in the space $\bar{V}$ such that

$$| g(x) - g(z)| < \in, \quad \forall x \in U.$$

We can write $U = \bar{V} \cap G$, where G is an open set in X. We have $G = (G \cap \bar{V}) \cup (G \setminus \bar{V})$. Now if $x \in G \cap \bar{V}$, then $f(x) = g(x)$ and so

$$|f(x) - f(z)| = |g(x) - g(z)| < \in.$$

If $x \in G \setminus \bar{V} \subset X \setminus \bar{V}$, $f(x) = 1$ and so

$$| f(x) - f(z)| = |1 - 1| = 0 < \in.$$

Thus $| f(x) - f(z)| < \in, \ \forall x \in G$. Hence f is continuous at z.

Since $x_0 \in A$(which is contained in $\bar{V}$) and $F \subset X \setminus \bar{V}$, we have $f(x_0) = 0$ and $f(x) = 1, \ \forall x \in F$. This proves that the space X is completely regular.

Theorem : 4. A locally compact Hausdorff space is completely regular.

Proof. Follows from Theorems 2 and 3.

Exercise : Let (X, τ) be a T_2 space. Then the following statements are equivalent.

(i) X is locally compact.

(ii) For each $x \in X$ and each neighbourhood U of x there is a relatively compact open set V such that $x \in V \subset \bar{V} \subset U$.

(iii) For each compact set C and each open set U with $C \subset U$, there is a relatively compact open set V such that $C \subset V \subset \bar{V} \subset U$.

(iv) τ has a basis consisting of relatively compact open sets.

Solution : (i) ⇒ (ii) :

As in Theorem 2 we can show that $\exists$ an open set V such that $\bar{V}$ is compact (i.e. V is relatively compact) and $x \in V \subset \bar{V} \subset U$.

(ii) ⇒ (iii) : Suppose (ii) holds good. Let C be a compact set and U be an open set with $C \subset U$. Take any $x \in C$. Then $x \in U$ and by (ii) there is a relatively compact open set V_x suct that.

$$x \in V_x \subset \bar{V}_x \subset U.$$

Now $\{V_x \,;\, x \in C\}$ forms an open cover of C. Since C is compact, $\exists$ a finite number.

of open sets $V_{x_1}, V_{x_2}, ..., V_{x_n}$ s.t. $C \subset \bigcup_{i=1}^{n} V_{x_i}$. Write $V = \bigcup_{i=1}^{n} V_{x_i}$. Then V is an open set. Also

$\bar{V} = \bigcup_{i=1}^{n} \bar{V}_{x_i}$, being finite union of compact sets is also compact. Hence V is a relatively

compact set such that

$$C \subset V \subset \bar{V} \subset U.$$

(iii) ⇒ (iv) : Suppose (iii) holds good. Let $\mathscr{B}$ be the family of all relatively compact open sets. Let G be any open set and $x \in G$. Since $\{x\}$ is compact, by (iii) $\exists$ a relatively compact open set V (i.e., $V \in \mathscr{B}$) such that $x \in V \subset \bar{V} \subset G$. Hence $\mathscr{B}$ forms a basis of τ.

(iv) ⇒ (i) : Let $x \in X$. Since X is open by (iv) there is a relatively compact open set V such that $x \in V \subset \bar{V} \subset X$. Clearly $\bar{V}$ is a compact neighbourhood of x and so X is locally compact.

Exercise : Let (X, τ) be locally compact and let $f : (X, \tau), \to (Y, \tau')$ be open, continuous and onto. Then show that Y is also locally compact.

Solution : Let $y \in Y$ and let V be a τ'-open set containing y. Let $f(x) = y$, $x \in X$. Since f is continuous at x, we can find an open set U containing x such that $f(U) \subset V$. By local compactness of (X, τ), there is a compact set A such that

$$x \in A^{\circ} \subset A \subset U.$$

Then $y = f(x) \in f(A^\circ) \subset f(A) \subset f(U) \subset V$.

Write $f(A) = B$. Since f is continuous and A is compact, so B is also compact. Again as f is open so $f(A^\circ)$ is an open set contained in $f(A)$ and so

$$f(A^\circ) \subset (f(A))^\circ = B^\circ.$$

Thus we have $y \in B^\circ \subset B \subset V$; which shows that Y is locally compact.

Exercise : Prove that a closed subspace of a locally compact space is locally compact.

Solution : Let (X, τ) be locally compact and let $Y \subset X$. To show that (Y, τ_Y) is so, choose $y \in Y$ and let V be a τ_Y-neighbourhood of y. Then $V = U \cap Y$ for some τ-neighbourhood U of y. Since X is locally compact, so there is a compact set A such that

$$y \in A^\circ \subset A \subset U.$$

Then $y \in A^\circ \cap Y \subset A \cap Y \subset U \cap Y = V$.

Write $B = A \cap Y$. Choose an open cover $\mathscr{V}$ of τ_Y-open sets covering B. Note that every $W \in \mathscr{V}$ is of the form $W = W' \cap Y$, $W' \in \tau$. Then $\{W' : W = W' \cap Y \in \mathscr{V}\} \cup (X \setminus Y)$ forms an open cover of A. By compactness of A, this cover has a finite subcover and consequently $\mathscr{V}$ also has a finite subcover of B. Hence B is τ_Y-compact. Clearly

$$y \in B^\circ \subset B \subset V$$

and this proves the result.

Exercise : The cartesian product $\prod_{\alpha \in \Lambda} X_\alpha$ (provided non-empty) is locally compact iff each X_α is locally compact $\forall \alpha \in \Lambda$ and all X_α, except for finite number of spaces, are compact.

Solution : First let X be locally compact. Since each projection map $p_\alpha : X \to X_\alpha$ is a continuous, open surjection, (by previous exercise) X_α is locally compact, $\forall \alpha \in \Lambda$.

Now let $x \in X$. By local compactness of X, x has a compact neighbourhood $U = \prod U_\alpha$ (say). Then $U_\alpha = X_\alpha$, $\forall \alpha \in \Lambda \setminus F$, where F is a finite subset of $\wedge$. Thus $p_\alpha(U) = X_\alpha$ which is compact (being continuous image of the compact set U), for all $\alpha \in \Lambda \setminus F$. Hence all spaces X_α, except for finite number of $\alpha's$, are compact.

Conversely, to prove X to be locally compact (under the stated conditions), let $x = (x_\alpha)_{\alpha\in\Lambda} \in X$. By hypothesis, there is a finite subset $F = \{\alpha_1, ..., \alpha_m\}$ (say) of Λ such that X_α is compact, $\forall \alpha \in \Lambda \setminus F$. For $\alpha_i \in F$ (i = 1, 2, ..., m), there exists a compact neighbourhood U_{α_i} of X_{α_i} in X_{α_i} (by local compactness of each X_α). Then $V = \prod_{\alpha\in\Lambda} V_\alpha$, where $V_\alpha = U_\alpha$ for $\alpha = \alpha_1, ..., \alpha_m$ and $V_\alpha = X_\alpha$ for $\alpha \notin F$ is a neighbourhood of x, and V is compact by Tychonoff product theorem. Hence each point of X has a compact neighbourhood and so X is locally compact.

2.2 Compactification

Let X be a topological space. A pair (f, Y) is said to be a compactification of X if the following conditions hold.

(i) Y is a compact space.

(ii) There is a subspace Y_0 of Y such that Y_0 is dense in Y and f is a homeomorphism of X onto Y_0.

Let (f, Y) be a compactification of the topological space X. If $Y\backslash f(X)$ consists of one point only, (f, Y) is called a one point compactification of the space X.

Exercise : 1. If X is a compact space, (i_X, X) is a compactification of X where i_X is the identity mapping.

Exercise : 2. If $X = (0, 1)$, then (f, Y) is a compactification of X where $Y = [0, 1]$ and $f : (0, 1) \to (0, 1) \subset [0, 1]$ is the inclusion.

Exercise 3. Take $Y = [a, b]$, $Y_0 = (a, b)$. Then also (f, Y) is a compactification of $(0, 1) = X$, where $f : X \to Y_0$ is defined by $f(x) = a + (b - a)x$, $\forall x \in X$.

Theorem : 1. Let (X, τ) be a non-compact topological space and let $X^* = X \cup \{\infty\}$, where ∞ is an element not in X. Denote by τ^* the family consisting of the void set ϕ, the set X^*, the members of τ and all those subsets U of X^* such that $X^*\backslash U$ is a closed compact subset of X. Then τ^* is a topology on X^* and (X^*, τ^*) is a compactification of X.

Proof : We prove the theorem in the following steps.

(I) We first verify that τ^* is a topology on X^*. Let G_1, G_2 be two members of τ^* and let $G = G_1 \cap G_2$. If $G = \phi$ then clearly $G \in \tau^*$. Suppose that $G \neq \phi$. If $G_1, G_2 \in \tau$ then $G \in \tau \subset \tau^*$. Let $G_1, G_2 \notin \tau$. Then $X^* \setminus G_1$ and $X^* \setminus G_2$ are closed compact subsets of X. So $X^* - G = (X^* - G_1) \cup (X^* - G_2)$ is also a closed compact subset of X and so $G \in \tau^*$. Again let $G_1 \in \tau$ but $G_2 \notin \tau$. Then $X^* \setminus G_2 = F$(say) is a closed compact subset of X.

Since $F \subset X$, $\infty \notin F$. So $\infty \in G_2$ and we may write

$G_2 = X^* \setminus F = \{\infty\} \cup (X \setminus F) = \{\infty\} \cup W$ (say) where $W \in \tau$. Then $G = G_1 \cap G_2 = G_1 \cap [\{\infty\} \cup W] = G_1 \cap W \in \tau \subset \tau^*$. If $G_1 \notin \tau$ and $G_2 \in \tau$ then one can similarly show that $G \in \tau^*$.

Now let $\{G_a \,;\, a \in A\}$ be any nonempty subfamily of τ^* (where A is an index set) and let $G = \cup \{G_a \,;\, a \in A\}$. If $G_a \in \tau \ \forall a \in A$, then clearly $G \in \tau \subset \tau^*$. Suppose $G_a \notin \tau$ for some $a \in A$. Let $A_1 = \{a \,;\, G_a \in \tau\}$ and $A_2 = A \setminus A_1$. Write $U_1 = \cup\{G_a \,;\, a \in A_1\}$ and $U_2 = \cup\{G_a \,;\, a \in A_2\}$. For $a \in A_2$, we may write $G_a = \{\infty\}\ U\ W_a$ where $W_a \in \tau$. Also write $W_a = G_a$ if $a \in A_1$. Then

$$
\begin{aligned}
G &= \cup\{G_a \,;\, a \in A\} \\
&= \{\infty\} \cup [\cup \{W_a \,;\, a \in A\}] \\
&= \{\infty\} \cup W \text{ (say).}
\end{aligned}
$$

clearly $W = \cup\{W_a \,;\, a \in A\} \in \tau$.

Take $a_0 \in A_2$. Then $W_{a_0} \subset W$. Now we have $X^* \setminus G = X^* \setminus [\{\infty\} \cup W] = X \setminus W$ which is closed is X. Also $X \setminus W \subset X \setminus W_{a_0} = X^* \setminus G_{a_0}$ where $X^* \setminus Ga_0$ is compact in X. Hence it follows that $X^* \setminus G$ is also compact in X (being a closed subset of a compact set). Therefore $G \in \tau^*$. Obviously ϕ, $X \in \tau^*$ and τ^* is a topology on X^*.

(II) Let $W \in \tau$. Then $\infty \notin W$. So we may write $W = X \cap W$ where $W \in \tau^*$. Again if $G \in \tau^*$ but $G \notin \tau$ then G is of the form $G = \{\infty\} \cup W$, where $W \in \tau$ and so $G \cap X = W \in \tau$. This shows that τ consists of exactly all those sets of the form $X \cap G$ where $G \in \tau^*$. Hence (X, τ) is a subspace of (X^*, τ^*). Also since $G \cap X \neq \phi$ for every open set G containing ∞, so X is dense in X^*.

(III) Let $G = \{G_a \,;\, a \in A\}$ be any open cover of X^*. Then $\infty \in G_a$ for some $a \in A$. Let $A_1 = \{a \,;\, \infty \notin G_a\}$ and $A_2 = A \setminus A_1$. For $a \in A_2$ we may write $G_a = \{\infty\} \cup W_a$ where $W_a \in \tau$. Also write $G_a = W_a$ for $a \in A_1$. Take any $a_0 \in A_2$. Then $X^* \setminus G_{a_0} = X \setminus W_{a_0}$ is a compact subset of X. Clearly $\{W_a \,;\, a \in A\}$ is an open cover of $X \setminus W_{a_0}$. So there is a finite number of sets $W_{a_1}, W_{a_2}, .., W_{a_n}$ from the family such that

$$X^* \setminus G_{a_0} = X \setminus W_{a_0} \subset \bigcup_{i=1}^{n} W_{a_i} \subset \bigcup_{i=1}^{n} G_{a_i}.$$

So $X^* = \bigcup_{i=0}^{n} G_{a_i}$ and X^* is compact.

(IV) Take $Y = X^*$ and $Y_0 = X$. Define the mapping $f : X \to Y_0$ by $f(x) = x$ for $x \in X$. Then f is a homeomorphism of X onto Y_0. Therefore (f, Y) is a compactification of X.

Note : In the above theorem clearly f is the identity mapping i_X on X. The compactification (i_X, X^*) is called the Alexandroff's one point compactification.

Definition : If (f, Y) is a compactification of a topological space X where Y is T_2 then (f, Y) is called a T_2-compactification of X.

Theorem 2 : Let X be a topological space which is not compact. Then Alexandroff's one point compactification (i_X, X^*) of X is a T_2-compactification iff X is a locally compact T_2 space.

Proof : First suppose that (i_X, X^*) is a T_2-compactification. Then X^* is a T_2 space and so X is also a T_2-space.

Let $x \in X$. Then x and ∞ are two distinct points of the space X^*. So there are open sets G_1, G_2 in X^* such that $x \in G_1$, $\infty \in G_2$ and $G_1 \cap G_2 = \phi$. Since $\infty \notin G_1$, $G_1 \subset X$ and is open in X. Again $X^* \backslash G_2$ is closed and compact in X. Since $G_1 \subset X^* \backslash G_2$, so $X^* \backslash G_2$ itself is a compact neighbourhood of x in X. Thus every point of X has a compact neighbourhood and so X is locally compact.

Next suppose that X is locally compact and T_2. Let x, y be two distinct points of X^*. If $x, y \in X$ then Hausdorffness of X implies that there are two disjoint open sets G_1, G_2 in X containing x and y respectively. Since open sets of X are also open in X^* so the result follows. Now let $x \in X$ and $y = \infty$. Since X is locally compact, there is a compact neighbourhood U of x in the space X. Since X is a T_2-space, U is also closed in X. There is an open set G_1 in X with $x \in G_1 \subset U$. Take $G_2 = X^* \backslash U$. Then G_2 is open in X^* and $\infty \in G_2$. Also $G_1 \cap G_2 = \phi$. Hence the space X^* is a T_2-space.

Definition : Let (X, τ) be a topological space. If there exists a topological space $(Y, \hat{\tau})$ s.t. X is homeomorphic to a subspace Y_0 of Y, then we say that X can be embedded in the space Y.

Let (X, τ) be a topological space and $\mathscr{F}$ be a family of functions s.t. each function f in $\mathscr{F}$ is a mapping from X to a topological space Y_f. Denote by Y the product of the spaces Y_f i.e., $Y = \prod_{f \in \mathscr{F}} Y_f$.

Definition : Define the mapping $e : X \to Y$ as follows. For $x \in X$, $e(x)_f = f(x)$, where $e(x)_f$ denotes the f th component of $e(x)$, i.e., $p_f(e(x)) = f(x)$, $\forall x \in X$ so that $p_f \circ e = f$, $p_f : \Pi Y_f \to Y_f$ being the f th projection map. e is called the evaluation map. We say that the family $\mathscr{F}$ distinguishes points iff for any two distinct points $x, y \in X$, $\exists f \in \mathscr{F}$ s.t. $f(x) \neq f(y)$. We say that the family $\mathscr{F}$ distinguishes points from closed sets if for any closed set A in X and each point x in $X \backslash A$, there is a $f \in \mathscr{F}$ s.t. $f(x) \notin \overline{f(A)}$.

Embedding Lemma

Let (X, τ) be a topological space and $\mathscr{F}$ be a family of functions on X such that each f in $\mathscr{F}$ is a continuous mapping of X into a topological space Y_f. Then followings hold.

(a) $e : X \to Y = \prod_{f \in \mathscr{F}} Y_f$ is continuous.

(b) e is an open mapping onto $e(X)$ if $\mathscr{F}$ distinguishes points from closed sets.

(c) e is one-to-one iff $\mathscr{F}$ distinguishes points.

(d) e is a homeomorphism of X onto $e(X)$ if $\mathscr{F}$ distinguishes points as also points from closed sets.

Proof : (a) Let P_f denote the projection of the product space Y to the f th co-ordinate space Y_f. For $x \in X$, $(P_f o e)(x) = P_f(e(x)) = e(x)_f = f(x)$. Since each f is continuous, so each $P_f o e$ is continuous for each $f \in \mathscr{F}$. Therefore e is continuous.

(b) Suppose that $\mathscr{F}$ distinguishes points from closed sets. Let G be any open set in X and let $y \in e(G)$. Then $\exists x \in G$ such that $y = e(X)$. Since $\mathscr{F}$ distinguishes points from closed sets, there is a function f in $\mathscr{F}$ such that $f(x) \notin \overline{f(A)}$ where $A = X \backslash G$. Write $U_f = Y_f \setminus \overline{f(A)}$. Then U_f is open in Y_f and $P_f^{-1}(U_f)$ is open in the product space $\prod Y_f$. Therefore $W_y = P_f^{-1}(U_f) \cap e(X)$ is an open set in the subspace $e(X)$ of Y. We now show that

$$y \in W_y \subset e(G).$$

Since $x \notin A$, by our hypothesis $f(x) \notin \overline{f(A)}$ and so $f(x) \in Y_f \setminus \overline{f(A)} = U_f$. Since $P_f(e(x)) = f(x)$, so $y = e(x) \in P_f^{-1}(U_f)$. Hence $y \in W_y$. Next let $z \in W_y$. Then $z \in P_f^{-1}(U_f)$ and $z \in e(X)$. Clearly $z = e(u)$ for some $u \in X$. Now $e(u) \in P_f^{-1}(U_f) \Rightarrow P_f(e(u)) = e(u)_f = f(u) \in U_f \Rightarrow f(u) \notin \overline{f(A)} \Rightarrow u \notin A \Rightarrow u \in G \Rightarrow z = e(u) \in e(G)$. So $W_y \subset e(G)$. Thus $e(G)$ is a neighbourhood of y in the space $e(X)$. Since y is an arbitary point of $e(G)$, it follows that $e(G)$ is an open set in the space $e(X)$. Hence $e : X \to e(X)$ is an open mapping.

(c) Suppose that $\mathscr{F}$ distinguishes points of X. Take any two points x, y $(x \neq y)$ in

X. Then there is a function $f \in \mathscr{F}$ such that $f(x) \neq f(y)$, i.e., $e(x)_f \neq e(y)_f$. This gives that $e(x) \neq e(y)$. Hence e is one-to-one.

Next, suppose that e is one-to-one. Let x and y be two distinct points of X. Then $e(x) \neq e(y)$. So there is a function f in $\mathscr{F}$ suct that $e(x)_f \neq e(y)_f$ i.e. $f(x) \neq f(y)$. Hence the family $\mathscr{F}$ distinguishes points of X.

(d) If the family $\mathscr{F}$ distinguishes points as well as distinguishes points from closed sets, then by (a), (b) and (c), e is a bijective mapping from X onto $e(X)$ which is both open and continuous. Hence e is a homeomorphism from X onto $e(X)$.

Definition : Let X be a topological space. Denote by $C^*(X)$ the family of all continuous mappings of X into the unit closed interval $[0, 1] = Q$. Now by Tychonoff's theorem $Q^{C^*(X)}$ [the product of the unit interval Q taken $C^*(X)$ times] is compact. As before let $e : X \to Q^{C^*(X)}$ be the evaluation map defined by $e(x)_f = f(x)$ for $x \in X$. Then e is continuous. Now suppose that X is a Tychonoff space (completely regular T_1 space). Then from definition it follows that the family $C^*(X)$ distinguishes points of X as well as points from closed sets. Then by Embedding Lemma, e is a homeomorphism of X onto the subspace $e(X)$ of $Q^{C^*(X)}$. We write $\beta(X) = \overline{e(X)}$. Then $\beta(X)$ is compact and the pair $(e, \beta(X))$ is a compactification of X which is called the Stone-Cech compactification of X.

Theorem : 3. (Stone-Cech Theorem)

Let X be a Tychonoff space and f be a continuous mapping of X into a compact T_2 space Y. Then there is a continuous extension of f which carries $\beta(X)$ into Y.

Proof : Let e denote the evaluation map of X into $Q^{C^*(X)}$ and g be the evaluation map of Y into $Q^{C^*(Y)}$ where $Q = [0, 1]$. If $a \in C^*(Y)$, then aof is a continuous mapping of X into Q and so aof $\in C^*(X)$. Denote the mapping $f^* : C^*(Y) \to C^*(X)$ by $f^*(a) = a$of, for all $a \in C^*(Y)$. Then for any $q \in C^*(X) \to Q$, $q$of* is a mapping of $C^*(Y)$ into Q. Define the mapping $f^{**} : Q^{C^*(X)} \to Q^{C^*(Y)}$ by

$$f^{**}(q) = q\text{of}^* \text{ for all } q \in Q^{C^*(X)}.$$

Let $a \in C^*(Y)$ and $q \in Q^{C^*(X)}$. We have

$$P_a o f^{**}(q) = P_a(f^{**}(q)) = P_a(q o f^*)$$

$$= (q o f^*)(a) = q(f^*(a)).$$

But $q(f^*(a))$ is simply the projection of q into the $f^*(a)$-th co-ordinate space of $Q^{C^*(X)}$ and this is a continuous mapping. Hence the mapping f^{**} is continuous. By embedding Lemma, e is a homeomorphism of X onto $e(X)$ and g is a homeomorphism of Y onto $g(Y) = \beta(Y)$, because Y is a compact T_2 space.

$$\begin{array}{ccc} \beta(X) \subset Q^{C^*(X)} & \xrightarrow{\;f^{**}\;} & Q^{C^*(Y)} \supset \beta(Y) = g(Y) \\ e \uparrow & & \uparrow g \\ X & \xrightarrow{\;f\;} & Y \end{array}$$

Let $x \in X$ and $a \in C^*(Y)$. Write $q = e(x)$ and $y = f(x)$.

Then $q \in Q^{C^*(X)}$ and $y \in Y$. We have

$$[(f^{**} o e)(x)](a) = [f^{**}(e(x))](a) = [f^{**}(q)](a)$$

$$= (q o f^*)(a) = q(f^*(a)) = q(a o f)$$

$$= e(x)(a o f) = (a o f)(x) = a(f(x)) = a(y)$$

$$= g(y)_a = g(y)(a) = [g(f(x))](a)$$

This gives that

$$(f^{**} o e)(x) = g(f(x)) \in g(Y).$$

Let $q \in e(X)$. Then there is a point x in X such that $e(x) = q$. So

$$f^{**}(q) = f^{**}(e(x)) = (f^{**} o e)(x) = g(f(x)) \in g(Y).$$

Or, $(g^{-1} o f^{**})(q) = f(x) \in Y$..............(1)

Now let $q \in \beta(X) \setminus e(X)$. Since $e(X)$ is dense in $\beta(X)$, there is a net $\{q_n ; n \in D\}$ in $e(X)$ such that $\{q_n ; n \in D\}$ converges to q. Clearly $g^{-1} o f^{**}$ is continuous. So the net

$\{g^{-1}of^{**}) (q_n) ; n \in D\}$ converges to $(g^{-1}of^{**}) (q)$. Since $(g^{-1}of^{**}) (q_n) \in Y$, for every $n \in D$ and Y is a compact T_2-space, $(g^{-1}of^{**}) (q) \in Y$. Hence $g^{-1}of^{**}$ carries $\beta(X)$ into Y.

Write $h = foe^{-1}$. Let $q \in e(X)$. Then $q = e(x)$ for some $x \in X$. We have

$$h(q) = (foe^{-1}) (q) = f(e^{-1} (q)) = f(x) = (g^{-1}of^{**}) (q). \qquad \text{(by (1))}$$

Hence $g^{-1}of^{**}$ is the required extension of h.

Definition : Let X be a topological space. Let us denote by $\mathscr{C}$ the collection of all compactifications of X. For (f, Y) and (g, Z) in $\mathscr{C}$ we define $(f, Y) \geq (g, Z)$ if there is a continuous mapping h of Y onto Z such that hof = g. Clearly gof^{-1} is a homeomorphism of Y_0 onto Z_0. The compactifications (f, Y) and (g, Z) of X are said to be equivalent if there is a homeomorphism h of Y onto Z s.t. hof = g. In this case we write $(f, Y) = (g, Z)$.

Theorem : 4. Let X be a topological space. Denote by $\mathscr{C}$ the collection of all T_2-compactifications of X. Then $\mathscr{C}$ is partially ordered by $\geq$.

Proof : Clearly $\geq$ is reflexive. Let (f, Y), (g, Z) and (h, W) be three elements of $\mathscr{C}$ and let $(f, Y) \geq (g, Z)$ and $(g, Z) \geq (h, W)$. There are continuous functions $j : Y \to Z$ (onto) and $k : Z \to W$ (onto) suct that $g = j$of and h = kog. Then clearly $h = ko$(jof) = (koj)of, where koj is a constant mapping of Y onto W. So $(f, Y) \geq (h, W)$. Hence $\geq$ is transitive.

Next let (f, Y) and $(g, Z) \in \mathscr{C}$ such that $(f, Y) \geq (g, Z)$ and $(g, Z) \geq (f, Y)$. There are continuous mappings $j : Y \to Z$ (onto) and $k : Z \to Y$ (onto) such that $j$of = g and kog = f.

So $f = ko(jof) = (koj)$of

and $g = (jof)$ = (jok)og.

Let $y \in Y_0$. Since f is a homeomorphism of X onto Y_0, there is a point x in X suct that $y = f(x)$. We have

$y = f(x) = [(ko\, j)o\, f]\ (x)$ = (koj) $(f(x))$ = (koj) (y). Again let $y \in Y \backslash Y_0$. Since Y_0 is dense in Y, there is a net $\{y_n ; n \in D\}$ in Y_0 converging to y. For each n in D, there is a point x_n in X such that $y_n = f(x_n)$. We have

$y = \lim y_n = \lim f(x_n) = \lim [(ko\,j)of]\,(x_n)$

$= \lim (koj)\,(f(x_n)) = \lim (koj)\,(y_n) = (koj)\,(y)$ (koj is a continuous mapping of Y onto inself). Thus $(koj)\,(y) = y,\ \forall y \in Y$, which shows that koj is the identity mapping of Y. Similarly, we can show that jok is the identity mapping of Z. Let $u, v\ (u \neq v)$ be two elements of Y. Then

$$(koj)\,(u) \neq (koj)\,(v)$$

i.e., $k(j(u)) \neq k(j(v))$

which implies that $j(u) \neq j(v)$. So j is one-to-one, similarly, k is one-to-one. Hence j is a homeomorphism of Y onto Z. This gives that (f, Y) and (g, Z) are equivalent i.e., $(f, Y) = (g, Z)$. Hence $(\mathscr{C}, \geq)$ is partially ordered.

Theorem : 5. Let X be a Tychonoff space which is not compact. Let $\mathscr{C}$ denote the collection of all T_2-compactifications of X. Then Alexandroff's one point compactification X^* is the minimal element and Stone-Cech compactification is the maximal element of $(\mathscr{C}, \geq)$.

Proof : Let (f, Y) be any T_2-compactification of X. Then Y is a compact T_2-space and f is a homeomorphism of X onto a dense subspace Y_0 of Y.

(I) We first show that $(e, \beta(X)) \geq (f, Y)$. Let $Q = [0, 1]$ and e denote the evaluation map of X into the space $Q^{C^*(X)}$. By Stone-Cech theorem there is a continuous extension h(say) of the map foe^{-1} such that h carries $\beta(X)$ into the space Y.

Let $y \in Y$. If $y \in Y_0$, then $y = f(x)$, for some x in X. Write $q = e(x)$. Then

$h(q) = (foe^{-1})\,(q) = f(e^{-1}\,(q)) = f(x) = y.$

Suppose that $y \in Y \setminus Y_0$. Since Y_0 is dense in Y, there is a net $\{y_n\ ;\ n \in D\}$ in Y_0 which converges to y. For each n in D, there is a point x_n in X such that $y_n = f(x_n)$. Write $q_n = e(x_n)$. Then $q_n \in e(X) \subset \beta(X),\ \forall n \in D$. Since $\beta(X)$ is compact, the net $\{q_n\ ;\ n \in D\}$ has a convergent subnet. So we may assume that the net $\{q_n\ ;\ n \in D\}$ is convergent. Let $q = \lim q_n$. Since h is continuous, the net $\{h(q_n)\ :\ n \in D\}$ converges to $h(q)$.

We have

$$h(q_n) = (f \circ e^{-1})\ (q_n) = f(e^{-1}(q_n)) = f(x_n) = y_n.$$

So $y = \lim y_n = \lim h(q_n) = h(q)$.

Hence h maps $\beta(X)$ onto the space Y which proves that $(e, \beta(X)) \geq (f, Y)$.

(II) We now assume X to be locally compact also. Then (i_X, X^*) is a T_2-compactification. Further Y_0 is an open subset of the Hausdorff compact space Y as it is homeomorphic with X. We now show that $(f, Y) \geq (i_X, X^*)$. Define the mapping h of Y onto $X^* = X \cup \{\infty\}$ as follows. Let $y \in Y_0$. Then $y = f(x)$, for some x is X. Since f is one-to-one, x is uniquely determined by y. We define $h(y) = x$. Also we define $h(y) = \infty$ if $y \in Y \setminus Y_0$.

Let G be any open set in X^*. First suppose that $G \subset X$. Then $h^{-1}(G) = f(G)$. Since f is a homeomorphism, $h^{-1}(G)$ is open in Y_0 and so is open in Y. Next let $\infty \in G$. Then we can write $G = W \cup \{\infty\}$ where W is open in X and $X^* \setminus G = X \setminus W$ is a closed compact subset of X. We have

$$h^{-1}(X^* \setminus G) = h^{-1}(X^*) \setminus h^{-1}(G) = Y \setminus h^{-1}(G).$$

So $h^{-1}(G) = Y \setminus h^{-1}(X^* \setminus G) = Y \setminus h^{-1}(X \setminus W) = Y \setminus f(X \setminus W)$

Since $X \setminus W$ is compact in X and f continuous, $f(X \setminus W)$ is compact in Y_0 and so is compact in Y. Since Y is T_2, so $f(X \setminus W)$ is closed in Y. Thus $h^{-1}(G)$ is open in Y and so h is continuous. Hence $(f, Y) \geq (i_X, X^*)$. This completes the proof.

2.3 Wallman Compactification

We will now describe another type of compactification.

Let (X, τ) be a T_1-topological space. Denote by $\mathscr{C}$, the collection of all closed subsets of X. Let Y denote the collection of all ultrafilters of $\mathscr{C}$.

(I) For any $G \in \tau$, let

$$G^* = \{\mathscr{F} : \mathscr{F} \in Y \text{ and } A \subset G \text{ for some } A \in \mathscr{F}\} \text{ and } \mathscr{B} = \{G^* : G \in \tau\}.$$

If G_1 and G_2 are two members of τ, then we can verify that

(a) $(G_1 \cap G_2)^* = G_1^* \cap G_2^*$

(b) $(G_1 \cup G_2)^* = G_1^* \cup G_2^*$

The relation (a) gives that $G_1^* \cap G_1^* \in \mathscr{B}$ whenever $G_1^*, G_2^* \in \mathscr{B}$. Let $\mathscr{F} \in Y$. Since $X \in \tau$ and $A \subset X$ for any $A \in \mathscr{F}$, so $\mathscr{F} \in X^*$. Hence $Y = X^*$. Therefore $\mathscr{B}$ forms a basis for a topology τ^* on Y.

(II) For any $x \in X$, let $\mathscr{F}_x = \{A : x \in A \in \mathscr{C}\}$. It is easy to see that each $\mathscr{F}_x$ is an ultrafilter in $\mathscr{C}$ and so $\mathscr{F}_x \in Y$. Let

$$Y_0 = \{\mathscr{F}_x : x \in X\}.$$

(III) Now define the mapping $f : X \to Y_0$ by $f(x) = \mathscr{F}_x$ for $x \in X$. Obviously f is an one-to-one mapping from X onto Y_0.

Let $x_0 \in X$ and let S be any neighbourhood of $f(x_0) = \mathscr{F}_{x_0}$ in the space Y_0. Since $\mathscr{B}$ is a base for τ^*, there is a member $G^* \in \mathscr{B}$ with

$$\mathscr{F}_{x_0} \in G^* \cap Y_0 \subset S.$$

So there is a member A in $\mathscr{F}_{x_0}$ with $A \subset G$. This gives that $x_0 \in G$. Take any $x \in G$ and let $B = \{x\}$. Then $B \in \mathscr{F}_x$. Since $B \in G$, so $\mathscr{F}_x \in G^*$ which shows that $f(x) \in S$. Hence f is continuous.

Write $g = f^{-1}$. Take any $\mathscr{F}_0 \in Y_0$. Then $\mathscr{F}_0 = \mathscr{F}_{x_0}$, for some point $x_0 \in X$. We have

$f(x_0) = \mathscr{F}_{x_0} = \mathscr{F}_0$ and so $g(\mathscr{F}_0) = x_0$. Let G be any open neighbourhood of x_0 in (X, τ). Then G^* is an open neighbourhood of $\mathscr{F}_0$ in the space Y and $S = Y_0 \cap G^*$ is an open neighbourhood of $\mathscr{F}_0$ in the space Y_0. Take any $\mathscr{F} \in S$. Now $\mathscr{F} = \mathscr{F}_x$, for some $x \in X$. So $A \subset G$ for some $A \in \mathscr{F}_x$ which implies that $x \in G$. Since $g(\mathscr{F}) = g(\mathscr{F}_x) = x$, we have $g(\mathscr{F}) \in G$ for all $\mathscr{F} \in S$. Hence g is also continuous. Therefore f is a homeomorphism from X onto Y_0.

(IV) Let $\mathscr{F} \in Y$ and let S be any $\mathscr{F}^*$-neighbourhood of π in the space Y. Since $\mathscr{B}$ is a base for τ^*, $\mathscr{F} \in G^* \subset S$ for some G^* in $\mathscr{B}$. Take any $x \in G$. Then $\mathscr{F}_x \in G^*$ and $\mathscr{F}_x \in Y_0$.

Thus $S \cap Y_0 \neq \phi$, which proves that Y_0 is dense in Y.

(V) Finally we show that (Y, τ^*) is compact. Let $\mathscr{Y}$ be any ultrafilter in Y. We define a subfamily β of $\mathscr{C}$ as follows :

$$\beta = \{A : A \in \mathscr{C} \text{ and } F \subset Y \setminus U_A^*, \text{ for some } F \in \mathscr{Y}\}$$

where $U_A = X \setminus A$ and so $U_A^* \in \tau^*$.

It is easy to see that $\phi \notin \beta$.

Let A_1, A_2 be two members of β. There are members F_1 and F_2 in $\mathscr{Y}$ such that

$$F_1 \subset Y \setminus U_{A_1}^* \text{ and } F_2 \subset Y \setminus U_{A_2}^*.$$

Write $F = F_1 \cap F_2$. Then $F \in \mathscr{Y}$ and we have

$$F \subset (Y \setminus U_{A_1}^*) \cap (Y \setminus U_{A_2}^*) = Y \setminus (U_{A_1}^* \cup U_{A_2}^*)$$

$$= Y \setminus (U_{A_1} \cup U_{A_2})^* = Y \setminus U_{(A_1 \cap A_2)}^*.$$

Hence $A_1 \cap A_2 \in \beta$.

Again let $A \in \beta$, $B \in \mathscr{C}$ and $A \subset B$. Then $F \subset Y \setminus U_A^*$, for some $F \in \mathscr{Y}$. Since $A \subset B$, $U_B \subset U_A$, which implies $U_B^* \subset U_A^* \Rightarrow Y \setminus U_A^* \subset Y \setminus U_B^*$. So $F \subset Y \setminus U_B^*$. This shows that $B \in \beta$. Hence β is a filter on $\mathscr{C}$. Now there is an ultrafilter β^* in $\mathscr{C}$ containing β. Clearly $\beta^* \in Y$.

We will show that β is a cluster point of $\mathscr{Y}$. If not then there is an $F \in \mathscr{Y}$ such that $\beta^* \notin \overline{F}$, where $\overline{F}$ denotes the τ^*-closure of F. Then $\beta^* \in Y \setminus \overline{F}$. Since $\mathscr{B}$ is a basis of τ^*, there is a member G^* in $\mathscr{B}$ with

$$\beta^* \subset G^* \subset Y \setminus \overline{F}.$$

Write $A = X \setminus G$. Then $A \in \mathscr{C}$ and $U_A = X \setminus A = G$. So $U_A^* = G^*$. From above we have

$$U_A^* \subset Y \setminus \overline{F} \text{ or } \overline{F} \subset Y \setminus U_A^*.$$

This gives that $A \in \beta$ and so $A \in \beta^*$. Since $\beta^* \in G^*$, there is a member B in β^* with

$B \subset G = X \setminus A$. But then $A \cap B = \phi \in \beta^*$ which contradicts that β^* is a filter. Hence $\beta^* \in \bar{F}$, for every $F \in \mathscr{Y}$ and so $\mathscr{Y}$ converges to β^*. This shows that (Y, τ^*) is compact. We have then the following

Theorem : (f, Y) is a compactification of (X, τ). This compactification (f, Y) of X is known as Wallman compactification of X and is denoted by $\omega(x)$.

Exercise : Let X be a Tychonoff space. If X is locally compact then prove that for every compactification (f, Y) of X, $Y \setminus f(X)$ is closed.

Solution : Since local compactness remains invariant under a homeomorphism, $f(X)$ is also locally compact. Let $y \in f(X)$. We will show that y is an interior point of $f(X)$. Since $f(X)$ is locally compact, $\exists$ an open neighbourhood U of x in $f(X)$ such that $\tilde{U}$ is compact ($\tilde{A}$ denotes the closure in $f(X)$). Hence there is an open set V in Y such that $U = V \cap f(X)$. Now we have $\bar{V} = \overline{(V \cap f(X))}$, where $\bar{A}$ denotes the closure of A in Y [as we know that if D is dense in a topological space (X, τ) then for any open set W, $\bar{W} = \overline{(W \cap D)}$].
Thus

$$x \in V \subset \bar{V} = \overline{(V \cap f(X))} \subset \bar{U} \subset \tilde{U} \subset f(X)$$

because $\tilde{U}$ is a closed set in $f(X)$ containing U. This proves the assertion.

Group–A
(Short questions)

1. Give an example of a topological space which is locally compact but not compact.
2. Show that the space of rationals with the induced topology from the usual topology of reals in not locally compact.
3. Give examples to justify that two compactifications of a given topological space may not be homeomorphic.
4. Show that any open subspace of a locally compact space is locally compact.

5. Prove that R^{ω} with product topology is not locally compact.
6. Show that if X is connected then $\beta(X)$ is also connected.
7. Show that product of locally compact spaces may not be locally compact.
8. For two compactifications (f, Y) and (g, Z) if $(f, Y) \leq (g, Z)$ and $(g, Z) \leq (f, Y)$, then show that (f, Y) and (g, Z) are equivalent.
9. Show that the Sorgenfrey line $(\mathbb{R}, \tau_l)$ in not locally compact at any point.

Group–B
(Long questions)

1. In order that two compactifications (f, Y) and (g, Z) of a topological space X be equivalent prove that it is necessary and sufficient that for every pair of closed subsets A, B of X,
$$\overline{f(A)} \cap \overline{f(B)} = \phi \Leftrightarrow \overline{g(A)} \cap \overline{g(B)} = \phi .$$
2. Show that if (f_α, Y_α) is a compactification of the space X_α for every $\alpha \in \Delta$ then $\prod_{\alpha\in\Delta} (f_\alpha, Y_\alpha)$ is a compactification of $\prod_{\alpha\in\Delta} X_\alpha$.
3. Let X be a Tychonoff space. Then prove that every pair of sets which can be separated by a real valued continuous function have disjoint closures in βX.
4. Let X be a Tychonoff space. Prove that X is locally compact iff the remainder $\beta X \backslash \beta(X)$ is closed.
5. Prove that in a locally compact space the intersection of a closed subset with an open subset in also locally compact.
6. With reasons give an example of a topological space which has only one compactification.
7. Let X be completely regular and T_1. Show that X is connected if and only if βX is connected.
8. Let X be discrete. Show that if U is open in βX then $\overline{U}$ is also open in βX. Then show that βX is totally disconnected.
9. Let A be any subset of $\mathbb{R}^n$ $(n \in \mathbb{N})$ such that A and $\mathbb{R}^n \backslash A$ are both dense in $\mathbb{R}^n$. Prove that no point of A has a compact neighbourhood.

Unit–III ❑ Paracompactness

Introduction

In this chapter we will study a weaker notion of compactness, which is called para-compactness ; this notion is actually more recent as it was introduced in 1944. The notion uses the idea of locally finite family which is easier to find. The importance of the idea of paracompactness also lies in the fact that many results, especially involving separation axioms on which we will concentrate, which were originally proved using compactness can be found valid by using the weaker notion of paracompactness. Apart from establishing several basic results we will show that a stronger notion of normality, called fully normal spaces can be obtained from paracompactness. We will finally introduce the very important notion of partition of unity and give an equivalent criteria of paracompactness in respect of partition of unity.

Definition : A family of subsets $\{B_i : i \in \Delta\}$ of a topological space X is said to be locally finite if each point x of X has a neighbourhood U which intersects at most finite number. of members of the family i.e., there is a finite subset Δ_1 of Δ suct that $U \cap B_i = \phi$ for all i in $\Delta \backslash \Delta_1$. Thus $\{B_i : i \in \Delta\}$ is locally finite iff there is an open cover α of X such that every member of α meets at most finite number of members of $\{B_i : i \in \Delta\}$.

Remark Every finite family is clearly locally finite. Every subfamily of a locally finite family is so. If $\{B_i : i \in \Delta\}$ is locally finite and $\{C_i : i \in \Delta\}$ is such that $C_i \subset B_i$, $\forall\, i$, then $\{C_i : i \in \Delta\}$ is locally finite.

Lemma 1. Let $\{B_i : i \in \Delta\}$ be a locally finite family of subsets in a topological space X. Then $\{\overline{B_i} : i \in \Delta\}$ is also locally finite and $\cup \{\overline{B_i} : i \in \Delta\} = \overline{\cup \{B_i : i \in \Delta\}}$.

Proof : Let $x \in X$. Then there is an open neighbourhood U of x and a finite subset Δ_1 of Δ such that $U \cap B_i = \phi$, $\forall\, i \in \Delta \backslash \Delta_1$. Then $U \cap \overline{B_i} = \phi$ $\forall\, i \in \Delta \backslash \Delta_1$ and so $\{\overline{B_i} : i \in \Delta\}$

is also locally finite. Now for each i, since $B_i \subset \cup\{B_i : i\in\Delta\}$, so $\overline{B_i} \subset \overline{\cup\{B_i : i\in\Delta\}}$ and hence $\cup\{\overline{B_i} : i\in\Delta\} \subset \overline{\cup\{B_i : i\in\Delta\}}$. Again let $x\in\overline{\cup\{B_i : i\in\Delta\}}$. Now there are a neighbourhood V of x and a finite subset Δ_1 of Δ such that $V \cap B_i = \phi$, $\forall\, i\ \Delta\backslash\Delta_1$. Let W be any neighbourhood of x. Then $V \cap W$ is also a neighbourhood of x and $(V \cap W) \cap (\cup\{B_i : i\in\Delta\backslash\Delta_1\}) = \phi$. Since $(V \cap W) \cap (\cup\{B_i ; i\in\Delta\}) \neq \phi$, so $(V\cap W) \cap (\cup\{B_i : i\in\Delta_1\}) \neq \phi$. Thus $x\in\overline{\bigcup_{i\in\Delta_1} B_i} = \cup\{\overline{B_i} : i\in\Delta_1\} \subset \cup\{\overline{B_i} : i\in\Delta\}$. This proves the result.

Definition 1. A topological space (X, τ) is said to be paracompact if every open cover of X has a locally finite refinement which is also an open cover of X.

Exercise 1. A compact space is paracompact.

Exercise 2. A discrete space is paracompact.

Theorem 1. Every closed subset of a paracompact space is paracompact.

Proof : Let X be paracompact and $F \subset X$ be closed. Let $\{U_i : i\in\Delta\}$ be an open cover of F. Then for each $i\in\Delta$, $\exists$ an open set V_i in X such that $U_i = V_i \cap F$. Now $\{V_i : i\in\Delta\} \cup \{X\backslash F\}$ is an open cover of X. Since X is paracompact, this open cover has a locally finite refinement α which is also an open cover of X. Then $\beta = \{U \cap F : U \in\alpha\}$ is clearly of locally finite cover of F. Evidently every member of β is open in F and is contained in some U_i. Hence F is paracompact.

Theorem : 2. If every open cover of a topological space X has a closed locally finite refinement then X is paracompact.

Proof : Let α_1 be any open cover of X and let α_2 be a closed locally finite refinement of α_1 covering X. Then for each $x\in X$, there is an open neighbourhood P_x of x which meets at most finite number of members of α_2. Now $\{P_x : x\in X\} = \alpha_3$ (say) is an open cover of X. So $\exists$ a closed locally finite refinement α_4 of α_3 covering X. For each B in α_2, let U_B be a member of α_1 containing B and let V_B be the union of all those members of α_4 which are disjoint from B. By Lemma 1, V_B is a closed subset of X. Put $B^* = U_B \cap (X\backslash V_B)$ and denote by α_5 the class of all sets of the form B^*. Clearly each B^* is open in X. Since $B \subset B^*$, $\forall B \in \alpha_2$ and α_2 covers X, so α_5 also covers X. Evidently from definition, α_5 is a refinement of α_1.

We shall now prove that α_5 is locally finite. For each x in X, there is a neighbourhood Q_x of x which intersects at most a finite number of members of α_4 say $D_1, .., D_m$, because α_4 is locally finite. For $i = 1, 2, .., m$, D_i is contained in a member, say, P_{x_i} of α_3 ($\because$ α_4 is a refinement of α_3). For $i = 1, 2, .. m$, P_{x_i} meets at most a finite number of members of α_2, say, $Bi_1, .., Bi,n_i$. So D_i meets, if at all, these n_i members alone of α_2. If B is a member of α_2 which is different from all the sets B_{ij} ($i = 1, 2, .., m, j = 1, .., n_i$), then B is disjoint from all D_i ($i = 1, .., m$) and so B^* is also disjoint from all D_i ($i = 1, .., m$) ($\because \cup D_i \subset V_B$). Since Q_x meets at most the sets D_i ($i = 1, .., m$) and α_4 covers X, $Q_x \subset \cup\{D_i : i = 1 \text{ to } m\}$. Consequently Q_x does not meet B^*. Thus x has a neighbourhood Q_x which meets at most a finite number of numbers of α_5. Hence α_5 is an open locally finite refinement of α_1 covering X and so X is paracompact.

Theorem : 3. If for each open cover of a regular space X there is a locally finite refinement covering X, then for each open cover of X there is a closed locally finite refinement covering X.

Proof : Let α be any open cover of X. For each x in X $\exists$ a $A_x \in \alpha$ such that $x \in A_x$. Since X is regular, $\exists$ an open set B_x such that $x \in B_x \subset \overline{B_x} \subset A_x$. Now $\beta = \{B_x ; x \in X\}$ is an open refinement of α covering X. By our assumption, $\exists$ a locally finite refinement ν of β covering X. Let $\delta = \{\overline{B} : B \in \nu\}$. By Lemma 1, δ is locally finite. Since each $B \in \nu$ is contained in some B_x and $\overline{B_x} \subset A_x \in \alpha$, so δ is a refinement of α. Thus δ is a closed locally finite refinement of α covering X.

Definition : A family of subsets of a topological space is said to be σ-locally finite if it is the union of a countable number of locally finite families.

Theorem : 4. Every open σ-locally finite cover of a topological space has a locally finite refinement.

Proof : Let α be a σ-locally finite cover of a topological space X. Now α is the union of the countable family $\{\alpha_n : n \in N\}$ of locally finite open classes α_n in X. Put B_1

$= \phi$, $B_n = \bigcup_{1 \le m < n} \left(\bigcup_{A \in \alpha_m} A \right)$ for $1 < n \in N$ and denote by β the class of all subsets of A of the form $V \setminus B_n$; where $n \in N$ and $V \in \alpha_n$. β is evidently a refinement of α. Let $x \in X$. Let n be the least positive integer such that x belong to some W in α_n. Then $x \in W \setminus B_n$ and so β covers X. Moreover W is an open neighbourhood of x which is disjoint from all members of β of the form $V \setminus B_p$ for all $p > n$. Since for each $q \in N$, α_q is locally finite, for each positive integer $m \le n$, there is a neighbourhood U_m of x which intersects at most a finite number of members of α_m. Consequently the neighbourhood $\cap\{U_m : 1 \le m \le n\} \cap W$ of x meets at most a finite number of members of β. This proves the theorem.

Lemma 2. If every open cover of a regular space X has an open σ-locally finite refinement covering X then X is paracompact.

Proof : Follows from The 2, 3, 4.

Corollary 1. Every regular Lindeloff space is paracompact.

Theorem : 5. Every Hausdorff paracompact space is regular.

Proof : Let X be a T_2 paracompact space. Let F be a closed subset of X and $a \in X \setminus F$. For each x in F, there is an open neighbourhood N_x of x such that $a \notin \overline{N}_x$. Since X is paracompact, the open cover $\{N_x : x \in F\} \cup \{X \setminus F\}$ of X has an open locally finite refinement α covering X. Let β be the class of all those members of α which meet F. Then β, as a subclass of α, is locally finite. By Lemma 1, $\{\overline{B} : B \in \beta\}$ is locally finite. Also $\cup \{\overline{B} : B \in \beta\}$ is a closed set in X. Put $U = \cup\{B : B \in \beta\}$ and $V = X \setminus \cup \{\overline{B} : B \in \beta\}$. Then U, V are disjoint open subsets of X and $F \subset U$. Since β consists of all those members of α which meet F, for each B in β there is an x in F such that $B \subset N_x$. Now $\overline{B} \subset \overline{N}_x$ and $a \notin \overline{N}_x$. Hence for each $B \in \beta$, $a \in X \setminus \overline{B}$. Thus $a \in \cap \{X \setminus \overline{B} : B \in \beta\} = V$. Hence X is regular.

Exercise : Let X be a paracompact space and let A, B be two disjoint closed subsets. If for every $x \in B$ there exist open sets U_x, V_x such that $A \subset U_x$, $x \in V_x$, $U_x \cap V_x = \phi$ then there are open sets U, V such that $A \subset U$, $B \subset V$ and $U \cap V = \phi$.

Solution : Clearly the family of open sets $\{X\backslash B\} \cup \{V_x\}_{x\in B}$ forms an open covering of X. Since X is paracompact, this open cover has an open locally finite refinement $\{W_s\}_{s\in\Delta}$. Let

$\Delta_1 = \{s\in\Delta : W_s \subset V_x \text{ for some } x\in B\}$.

Then $A\cap\overline{W}_s = \phi$ for $s\in\Delta_1$ and $B \subset \bigcup_{s\in\Delta_1} W_s$. We know that $\bigcup_{s\in\Delta_1}\overline{W}_s = \overline{\bigcup_{s\in\Delta_1} W_s}$ which is closed. Consequently $U = X\backslash \bigcup_{s\in\Delta_1}\overline{W}_s$ is open. Evidently $A \subset U$, $B \subset V = \bigcup_{s\in\Delta_1} W_s$ and $U \cap V = \phi$.

Theorem : 6. Every paracompact Hausdorff space is normal.

Definition : Let $x\in X$, $B \subset X$ and α be a class of subsets of X. Then $\cup\{A \in\alpha : x \in A\}$ is called the star of x over α denoted by st(x, α). $\cup\{A\in\alpha : A \cap B \neq \phi\}$ is called the star of B over α and is denoted by st(B, α). The class of sets $\{st(x, \alpha) : x\in X\}$ is called the star of α.

Exercise : For $B \subset X$ and a class of subsets α of X,

(i) st $(B, \alpha) = \cup$ {st $(x, \alpha) : x \in B\}$

(ii) $B \subset \cup \{A : A\in\alpha\} \Rightarrow B \subset$ st (B, α).

Solution : (i) For every $x \in B$ clearly st $(x, \alpha) \subset$ st(B, α) and so $\cup$\{st$(x, \alpha) : x\in B\}$ $\subset$ st(B, α). Conversely, let $y\in$ st(B, α). Then from definition there is some $A\in\alpha$ such that $y\in A$ and $A\cap B \neq \phi$. Choose $x\in A \cap B$. Then $y\in A \subset$ st(x, α) and so st$(B, \alpha) \subset \cup$ {st$(x, \alpha) : x\in B\}$. This proves the result.

(ii) Let $y\in B$. Since $B \subset \cup\{A : A\in\alpha\}$ so $\exists$ a $A_1\in\alpha$ such that $y\in A_1$. Then $A_1\cap B \neq \phi$ and this imlies $y\in A_1 \subset \cup \{A\in\alpha : A \cap B \neq \phi\} =$ st(B, α). This is true for every $y\in B$ and so $B \subset$ st(B, α).

Exercise : For B, $C \subset X$ and a collection α of subsets of X.

(i) $B \subset C \Rightarrow$ st$(B, \alpha) \subset$ st(C, α).

(ii) $B \cap$ st$(C, \alpha) = \phi \Leftrightarrow C \cap$ st$(B, \alpha) = \phi$.

Solution : (i) is obvious.

(ii) First suppose that $B \cap \mathrm{st}(C, \alpha) \neq \phi$. Then $\exists\, y \in B \cap \mathrm{st}(C, \alpha)$. Since $\mathrm{st}(C, \alpha) = \cup\{A \in \alpha : A \cap C \neq \phi\}$, we can find a $A_1 \in \alpha$ such that $y \in A_1$ where $A_1 \cap C \neq \phi$. But then $y \in A_1 \cap B$ which implies $A_1 \subset \mathrm{st}(B, \alpha)$. Then $\mathrm{st}(B, \alpha) \cap C \neq \phi$. Similarly we can show that $C \cap \mathrm{st}(B, \alpha) \neq \phi \Rightarrow B \cap \mathrm{st}(C, \alpha) \neq \phi$. This completes the proof.

Definition : Let α and β be two classes of subsets, of X. Then α is called a star refinement of β if the star of α is a refinement of β.

Lemma : 3. Let α and β be covers of X such that α is a star refinement of β. Then $\{\mathrm{st}(A, \alpha) : A \in \alpha\}$ is a refinement of the star of β.

Proof : Let $A \in \alpha$. Since α is a star refinement of β, for each $x \in A$, there is a member $B_x \in \beta$ such that $\mathrm{st}(x, \alpha) \subset Bx$. Let $x_0 \in A$ be fixed. For each $x \in A$, $B_x \subset \mathrm{st}(x_0, \beta)$, because $x_0 \in A \subset \mathrm{st}(x, \alpha) \subset B_x$. Hence

$$\mathrm{st}(A, \alpha) = \cup\, \{\mathrm{st}(x, \alpha) : x \in A\} \subset \cup\{B_x : x \in A\} \subset \mathrm{st}(x_0, \beta).$$

This completes the proof.

Definition : If for every $x \in X$ there exists $t_x \in \Delta_0$ such that $\mathrm{st}(x, \alpha) \subset B_{t_x} \in \beta = \{B_t\}_{t \in \Delta_0}$, then α is called a pointwise star refinement of β.

Lemma : 4. If an open covering $\alpha = \{U_s\}_{s \in \Delta}$ of X has a closed locally finite refinement then it also has an open pointwise star refinement.

Proof : Let $\alpha = \{U_s\}_{s \in \Delta}$ be an open covering of X. By our assumption α has a closed locally finite refinement $\{F_t\}_{t \in \Delta_0}$. For every $t \in \Delta_0$ let us denote by $s(t)$, a fixed index in Δ such that $F_t \subset U_{s(t)}$. Since $\{F_t\}_{t \in \Delta_0}$ is locally finite, evidently $\Delta_0(x) = \{t \in \Delta_0 : x \in F_t\}$ is finite for every $x \in X$. Then it follows that the set

$$V_x = \bigcap_{t \in \Delta_0(x)} U_{s(t)} \cap \Big(X \setminus \bigcup_{t \notin \Delta_0(x)} F_t\Big)$$

is an open set, for every $x \in X$. Clearly $x \in V_x$ and so $\beta = \{V_x\}_{x \in X}$ in as open cover of X. Now let us consider a point $x_0 \in X$ and choose an index $t_0 \in \Delta_0(x_0)$. From the above construction it follows that if $x_0 \in V_x$ then $t_0 \in \Delta_0(x)$ and $V_x \subset U_{s(t_0)}$. Hence $\mathrm{st}(x_0, \beta) \subset U_{s(t_0)}$ which proves that β is a pointwise star refinement of α.

Lemma : 5. If a covering $\alpha = \{A_s\}_{s\in\Delta}$ of an arbitrary set is a pointwise star-refinement of a covering $\beta = \{B_t\}_{t\in\Delta_0}$ which is a pointwise star refinement of a covering $\nu = \{C_w\}_{w\in\Delta_1}$, then α is a star refinement of ν.

Proof : Let us take a fixed $s_0 \in \Delta$ and for every $x \in A_{s_0}$ choose $t(x) \in \Delta_0$ such that

$$A_{s_0} \subset st(x, \alpha) \subset B_{t(x)}.$$

Then we have

$st(A_{s_0}, \alpha) = \cup\{st(x, \alpha) : x \in A_{s_0}\} \subset \bigcup_{x\in A_{s_0}} B_{t(x)}$. From above we can get that whenever we choose $x_0 \in A_{s_0}$ then $x_0 \in B_{t(x)}$ for every $x \in A_{s_0}$.

Hence $\bigcup_{x\in A_{s_0}} B_{t(x)} \subset \mathrm{st}(x_0, \beta)$. But then

$$\mathrm{st}\,(A_{s_0}, \alpha) \subset \mathrm{st}(x_0, \beta) \subset C_w$$

for some $w \in \Delta_1$. This completes the proof.

Lemma : 6. If every open covering of a topological space X has an open star refinement then every open cover of X has an open σ-locally finite refinement.

We omit the proof as it is too technical.

From the above three Lemmas we can find the following equivalent conditions for paracompactness using the star operation.

Theorem : 7. For a regular Hausdorff space X the following are equivalent.

(i) X is paracompact.

(ii) Every open cover of X has an open pointwise star refinement.

(iii) Every open cover of X has an open star refinement.

(iv) Every open cover of X has an open σ-locally finite refinement.

Definition : A topological space X is said to be fully normal if every open cover of X has an open star refinement.

Exercise : A fully normal space X is normal.

Solution : Let X be a fully normal space. Let B and C be two pairwise disjoint closed subsets of X. Now $\{X\backslash B, X\backslash C\}$ is an open cover of X and so it has an open star refinement α covering X. Let $B^* = \text{st}(B, \alpha)$ and $C^* = \text{st}(C, \alpha)$. Clearly B^* and C^* are open in X and $B \subset B^*$, $C \subset C^*$. We claim that $B^* \cap C^* = \phi$. On the contrary if $x \in B^* \cap C^*$ then $\exists\, M, N \in \alpha$ such that $x \in M \cap N$ where $M \cap B \neq \phi \neq N \cap C$. Then $\text{st}(x, \alpha)$ intersects both B and C and as such can't be contained in either $X\backslash B$ or $X\backslash C$. This contradicts that α is an open star refinement of $\{X\backslash B, X\backslash C\}$. Hence X is normal.

We will end the discussions with giving the idea of partition of unity without going into the full details of the proofs.

Definition : A family $\{f_s\}_{s\in\Delta}$ of continuous functions defined on a space X with values in $[0, 1]$ is called a partition of unity if $\sum_{s\in\Delta} f_s(x) = 1$ for every $x \in X$. This actually means that for a fixed point $y \in X$, at most countably many functional values $f_s(y)$ can be non-zero and clearly the infinite series $\sum_{i=1}^{\infty} f_{s_i}(y)$ is convergent with its sum 1 where $\{s_1, .., , ...\} = \{s \in \Delta : f_s(y) \neq 0\}$. A partition of unity $\{f_s\}_{s\in\Delta}$ is said to be locally finite if the covering $\{f_s^{-1}((0, 1])\}_{s\in\Delta}$ is locally finite. In this case for every point $y \in X$, there is a neighbourhood U_y and a finite set $\Delta_0 = \{S_1, S_2, .., S_n\} \subset \Delta$ such that $f_s(x) = 0$ for $x \in U_y$, $s \in \Delta\backslash\Delta_0$. Clearly $\sum_{i=1}^{n} f_{s_i}(x) = 1$ for $x \in U_y$. We say that the partition of unity $\{f_s\}_{s\in\Delta}$ is subordinate to the covering $\{A_t\}_{t\in\Delta_1}$ if the covering $\{f_s^{-1}((0, 1])\}_{s\in\Delta}$ is a refinement of $\{A_t\}_{t\in\Delta_1}$.

Lemma : 7. For each point finite open cover $\{U_s\}_{s\in\Delta}$ (a cover is called point finite if every point belong to only a finite number of members of the cover) of a normal space

X, there exists an open cover $\{V_s\}_{s\in\Delta}$ such that $\overline{V_s} \subset U_s \forall s \in \Delta$.

Proof is omitted.

Lemma : 8. If an open cover $\mathscr{V}$ of a T_2 topological space X has a partition of unity $\{f_s\}_{s\in\Delta}$ subordinated to it, then $\mathscr{V}$ has a locally finite open refinement.

Proof left as an exercise in Group–B.

Theorem : 8. For a T_1 topological space X, following are equivalent.

(i) X is paracompact.

(ii) Every open cover of X has a locally finite partition of unity subordinated to it.

(iii) Every open cover of X has a partition of unity subordinated to it.

Proof : First suppose X is paracompact. Let $\mathscr{U} = \{U_s\}_{s\in\Delta_1}$ be an open cover of X. Let $\mathscr{V} = \{V_s\}_{s\in\Delta}$ be an open and locally finite refinement of $\mathscr{U}$. Then by previous Lemmas, we can get a cover $\{W_s\}_{s\in\Delta}$ of X such that $\bar{W}_s \subset V_s \quad \forall s\in\Delta$. By Uryshon's Lemma, there exists a continuous function $g_s : X \to [0, 1]$ such that $g_s(x) = 1$ for $x \in \bar{W}_s$, $g_s(x) = 0$ for $x\in X\backslash V_s$. Since $\mathscr{V}$ is locally finite, the function $g = \sum_{s\in\Delta} g_s$ is well-defined. It is easy to verify that defining $f_s = g_s/g \ \forall s\in\Delta$, the family $\{f_s\}_{s\in\Delta}$ is a locally finite partition of unity subordinated to $\mathscr{U}$. This proves (ii).

The implication (ii) $\Rightarrow$ (iii) is obvious. Let (iii) hold. In view of preceding Lemma we only need to prove that X is T_2. We will show that X is Tychonoff. Let $x\in X$, F–a closed set such that $x\notin F$. Now $\mathscr{V} = \{X\backslash F, X\backslash\{x\}\}$ is an open cover of X and so it has a partition of unity $\{f_s\}_{s\in\Delta}$ subordinated to it. So $\exists\ s_0\in\Delta$ such that $f_{s_0}(x) = a > 0$ and $f_{s_0}^{-1}((0, 1]) \not\subset X\backslash\{x\}$ i.e:, it is contained in $X\backslash F$. So $f_{s_0}(F) = 0$. Then $f : X \to [0, 1]$ where

$f(x) = \min\left\{\frac{1}{a} f_{s_0}(x), 1\right\}$ is a continuous function with $f(x) = 1$, $f(F) = 0$.

Group–A

(Short questions)

1. Give an example of an open cover which is point finite but not locally finite.
2. Let $\mathscr{V} = \{(-n, n)\}_{n\in N}$. Is $\mathscr{V}$ locally finite? Answer with reasons.
3. Give an example of a paracompact space that is not compact.
4. Prove that a discrete topological space is paracompact.
5. Show that every locally finite family of non-empty subsets of a countably compact space is finite.
6. Prove that a T_1 space is normal if each finite open cover has an open star refinement.
7. Give an example of a paracompact space which is not Lindeloff.
8. Give an example of a paracompact space which is not countably compact.

Group–B

(Long questions)

1. Show that an F_σ-subset of a paracompact space is paracompact.
2. Prove that the cartesian product $X \times Y$ of a paracompact space X and a compact space Y is paracompact.
3. Prove that a paracompact countably compact space is compact.
4. Prove that a Lindeloff space is paracompact.
5. If an open cover v of a T_2 topological space X has a partition of unity $\{f_s\}_{s\in\Lambda}$ subordinated to it then show that v has a locally finite open refinement.
6. Let X be a T_2-space. If $\exists$ a countable open cover $\{U_n\}_n$ of X such that $\bar{U}_n \subset U_{n+1} \forall n$ and $\bar{U}_n$ is compact $\forall n$ then prove that X is paracompact.

7. A topological space X is called (a) metacompact if every open cover of X has an open point finite refinement, (b) countably paracompact if every countable open cover of X has an open, locally finite refinement.

   Prove that a paracompact space is metacompact as well as countably paracompact.

8. Prove that a countable paracompact space is countably metacompact [a space X is countabley metacompact if every countable open cover of X has an open, point finite refinement]

9. Prove that any closed subspace of a countably paracompact (resp. metacompact, countably metacompact) space X is respectively so.

10. Let τ_l be the lower limit topology on $\mathbb{R}$. Assuming that $(\mathbb{R}, \tau_l)$ is Lindeloff, prove that $(\mathbb{R}, \tau_l)$ is a paracompact space.

Unit–IV ❑ Metrization

Introduction

After coming across the two structures, namely, metric spaces and topological spaces it is evident that a metric space is a much stronger structure than a topological space with many more additional properties arising due to the presence of the distance function. We have already seen for example that while four types of compactness are different in a topological space (in general with no additional assumption), they become equivalent in a metric topology. So the natural question is that whether it is possible to get a given topology on a set X as the topology induced by some metric on that set. Metrization deals with this problem. Here we will see that there are metrizable topologies as also topologies which are not metrizable. Uryshon's metrization theorem is recalled here. Nagata-Smirnov theorem; though quite long and tricky with a very deep proof, is the milestone of metrization problems as it gives necessary and sufficient conditions for a space to be metrizable. Further, in the last section we include two very important theorems, namely, Arzela–Ascoli's theorem and Stone Weirstrass theorem.

3.1 Metrization of topological spaces

Definition : A topological space X is said to be metrizable if there exists a metric d on the set X that induces the given topology of X.

Since a metric space is inherently Hausdorff, normal, and it satisfies first axiom of countability, say the least, metrizability is a highly desirable property for a topological space. Before we prove our main results, we recall the following.

Definition : Let (X, d) be a metric space. A subset A of X is called bounded if there is a number M such that $d(x, y) \leq M \ \forall x, y \in A$. If A is bounded, the diameter of A is defined to be the number diam A = lub $\{d(a_1, a_2) : a_1, a_2 \in A\}$.

Boundedness is not a topological property and it only depends on the particular metric d.

Theorem A. Let (X, d) be a metric space. Define $\bar{d} : X \times X \to R$ by the equation $\bar{d}(x, y) = \min\{d(x, y), 1\}$. Then $\bar{d}$ is a bounded metric that induces the topology of (X, d), irrespective of whether d is bounded or unbounded. $\bar{d}$ is called the standard bounded metric on X.

Lemma A. Let d and d' are two metrics on the set X and τ and τ' be their induced topologies. Then τ is finer than τ' iff for each x in X and each $\in > 0$, there is a $\delta > 0$ s.t.

$$B_d(x, \delta) \subset B_{d'}(x, \in)$$

Exercise : For any (+) ve integer n, R^n with the product topology is metrizable.

Solution : We shall prove that the euclidean metric d on R^n defined by

$$d(x, y) = [(x_1 - y_1)^2 + ... + (x_n - y_n)^2]^{1/2}$$

and the square metric defined by

$$\rho(x, y) = \max\{|x_i - y_i| : i = 1, 2, ..., n\}$$

for any $x = (x_1, .., x_n)$ and $y = (y_1, .., y_n)$

induce the product topology on R^n.

First note that for $x = (x_1, .., x_n)$, $y = (y_1, .., y_n) \in R^n$, we have

$$\rho(x, y) \leq d(x, y) \leq \sqrt{n}\, \rho(x, y)$$

which implies that for $x \in R^n$, and any $\in > 0$,

$$B_d(x, \in) \subset B_\rho(x, \in)$$

$$B_\rho(x, \in/\sqrt{n}) \subset B_d(x, \in).$$

Hence the two metric topologies induced by d and ρ are same.

Now we show that the product topology is the same as that induced by the metric ρ. Let $B = (a_1, b_1) \times .. \times (a_n, b_n)$ be a basis element of the product topology and let $x = (x_1, .., x_n) \in B$. For each i, $\exists$ an $\epsilon_i > 0$ s.t.

$$(x_i - \epsilon_i, x_i + \epsilon_i) \subset (a_i, b_i)$$

Choose $\epsilon = \min \{\epsilon_1, .., \epsilon_n\}$. Then $B_\rho(x, \epsilon) \subset B$. Hence the ρ-topology is finer than the product topology. That the product topology is also finer that the ρ-topology follows from the fact that for any $x = (x_1, .. x_n) \in R^n$ and $\epsilon > 0$,

$B_\rho(x, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times .. \times (x_n - \epsilon, x_n + \epsilon)$ is itself a member of the product topology. This completes the proof.

Definition : Given an index set J and given two points $x = (x_\alpha)_{\alpha\in J}$ and $y = (y_\alpha)_{\alpha\in J}$ of R^J, the uniform metric on R^J is defined by

$$\bar{\rho}(x, y) = \text{lub}\{\bar{d}(x_\alpha, y_\alpha)/\alpha \in J\}$$

where $\bar{d}$ is the standard bounded metric on R. The topology induced by the metric $\bar{\rho}$ is called the uniform topology.

Lemma : The uniform topology on R^J is finer than the product topology.

Proof left as an Exercise.

Theorem : The countable product of R, R^ω with the product topology is metrizable.

Proof : If $x = (x_n)_{n\in N}$ and $y = (y_n)_{n\in N}$ are two points of R^ω, define

$$D(x, y) = \underset{i}{\text{lub}}\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\}$$

where $\bar{d}$ is the standard bounded metric on R. Evidently $D(x, y) \geq 0$ and $= 0$ iff $x = y$. Also $D(x, y) = D(y, x)$. To prove the triangle inequality, we note that for $x, y, z \in R^\omega$

$$\frac{\bar{d}(x_i, z_i)}{i} \leq \frac{\bar{d}(x_i, y_i)}{i} + \frac{\bar{d}(y_i, z_i)}{i}$$

$$\leq D(x, y) + D(y, z) \quad \forall i \in N$$

and so $D(x, z) = \text{lub}\left\{\frac{\bar{d}(x_i, z_i)}{i}\right\} \leq D(x, y) + D(y, z)$. Thus D is a metric on R^ω. We shall now show that D induces the product topology on R^ω.

First let U be open in the metric topology and let $x \in U$. $\exists$ an $\epsilon > 0$ such that $B_D(x, \epsilon) \subset U$. Choose a (+) ve integer M s.t. $1/M < \epsilon$. Let

$V = (x_1 - \epsilon, x_1 + \epsilon) \times .. \times (x_M - \epsilon, x_M + \epsilon) \times R \times R \times \ldots$. Then V is open in the product topology. Note that for any $y = (y_i) \in R^\omega$,

$$\frac{\bar{d}(x_i, y_i)}{i} \leq \frac{1}{M} \quad \forall (i \geq M)$$

Hence $D(x, y) \leq \max \left\{\frac{\bar{d}(x_1, y_1)}{1}, .., \frac{\bar{d}(x_M, y_M)}{M}, \frac{1}{M}\right\}$. Clearly if $y \in V$ then $D(x, y) < \epsilon$ and so $V \subset B_D(x, \epsilon) \subset U$.

Conversely let $U = \prod_{i \in N} U_i$ be a basis element of the product topology, where U_i is open in R for $i = \alpha_1, .., \alpha_n$ and $U_i = R$ for all other indices i. Let $x \in U$. Choose an interval $(x_i - \epsilon_i, x_i + \epsilon_i) \subset U_i$ for $i = \alpha_1, .., \alpha_n$. We can choose each $\epsilon_i \leq 1$. Now take

$$\epsilon = \min \{\epsilon_i \setminus i \,;\, i = \alpha_1, .., \alpha_n\}.$$

If $y \in B_D(x, \epsilon)$, then $(\forall i)$

$$\frac{\bar{d}(x_i, y_i)}{i} \leq D(x, y) < \epsilon.$$

Clearly if $i = \alpha_1, .., \alpha_n$, then $\epsilon \leq \epsilon_i \setminus i$, so that $\bar{d}(x_i, y_i) < \epsilon_i \leq 1$ which in turn implies $|x_i - y_i| < \epsilon_i$. Therefore $y \in \prod U_i$ and so

$$x \in B_D(x, \epsilon) \subset U.$$

This completes the proof of the fact that D induces the product topology on R^ω.

Lemma : Let X be a toplogical space. Let $A \subset X$. If there is a sequence of points of A converging to x then $x \in \bar{A}$. The converse holds if X is metrizable.

This result is sometimes called the sequence Lemma. We shall use this result to conclude the following.

Exercise : An uncountable product of R with itself endowed with the product topology is not metrizable.

Proof : Let J be an uncountable index set. We show that R^J does not satisfy the sequence Lemma.

Let A be the subset of R^J consisting of all points (x_α) such that $x_\alpha = 0$ for finitely many values of α and $x_\alpha = 1$ for all other values of α. Let 0 be the point of R^J each of whose coordinates is 0.

Let ΠU_α be a basis open set in R^J containing 0. Then $U_\alpha \neq R$ for only finitely many values of α, say for $\alpha = \alpha_1, \ldots, \alpha_n$. Let (x_α) be the point defined by $x_\alpha = 0$ for $\alpha = \alpha_1, \ldots, \alpha_n$ and $x_\alpha = 1$ for all other values of α. Then clearly $x = (x_\alpha) \in A \cap \Pi U_\alpha$. This shows that $0 \in \overline{A}$.

But there is no sequence of points of A converging 0. For let $\{a_n\}$ be a sequence of points of A. Each point a_n is a point of the product space having only finitely many coordinates equal to 0. For $n \in N$, let J_n denote the subset of J consisting of those indices α for which the αth coordinate of a_n is zero. Then $\bigcup_{n \in N} J_n$ is a countable set. Since J is uncountable, there is at least one index, say, β s.t. $\beta \in J \setminus \cup J_n$. This means that βth coordinate of each a_n is equal to 1.

Now let $U_\beta = (-1, 1)$ and $U = \Pi_\beta^{-1}(U_\beta)$. Then U is an open nbd of 0 in R^J but $a_n \notin U$ for any $n \in N$. Thus the sequence $\{a_n\}$ cannot converge to 0.

The above example confirms that not every topological space is metrizable. We now give a few necessary and sufficient conditions for metrizability. First we recall the following well known theorem which provides some sufficient conditions for a topological space to be metrizable, for the proof of which we refer to your earlier course on set-topology.

Uryshon Metrization Theorem

Every regular T_1 and 2nd countable space X is metrizable.

Exercise : Let X be a regular space with a basis $\mathcal{B}$ that is σ-locally finite. Then X is normal.

Solution : Step 1.

Since $\mathcal{B}$ is σ-locally finite, we can write $\mathcal{B} = \bigcup_n \mathcal{B}_n$ where each $\mathcal{B}_n$ is locally finite. Let W be any open set in X. Let C_n be the collection of those basis elements B such that $B \in B_n$ and $\overline{B} \subset W$. Then C_n, being a subcollection of $\mathcal{B}_n$ is locally finite. Define

$$U_n = \bigcup_{B \in C_n} B$$

Then U_n is open and $\overline{U}_n = \bigcup_{B \in C_n} \overline{B}$ ($\because$ C_n is locally finite),

consequently $\cup U_n \subset \cup \overline{U}_n \subset W$.

We now show that $W = \cup U_n = \cup \overline{U}_n$. Let $x \in W$. By the regularity of X, $\exists$ a $B \in \mathcal{B}$ such that $x \in B \subset \overline{B} \subset W$. Now $B \in \mathcal{B}_n$ for some n. Then $B \in C_n$ by definition and so $x \in \overline{U}_n$.

Step 2. Now let C and D be two disjoint closed sets in X. By step 1 we can construct two countable collections of open sets $\{U_n\}$ and $\{V_n\}$ such that

$$\cup U_n = \cup \overline{U}_n = X \setminus D$$

$$\cup V_n = \cup \overline{V}_n = X \setminus C$$

For each $n \in \mathbb{N}$, define

$$U'_n = U_n \setminus \bigcup_{i=1}^{n} \overline{V}_i \text{ and } V'_n = V_n \setminus \bigcup_{i=1}^{n} \overline{U}_i.$$

and let $U' = \bigcup_{n=1}^{\infty} U'_n$ and $V' = \bigcup_{n=1}^{\infty} V'_n$

Then $U' \supset C$, $D \subset V'$ and U', V' are open sets with $U' \cap V' = \phi$.

Nagata Smirnov Metrization Theorem.

A topological space X is metrizable iff it is regular T_1 and has a σ-locally finite basis.

Proof : (sufficiency)

Let X be a regular T_1 space with a σ-locally finite basis $\mathscr{B}$.

Step 1. Let W be open in X. We have already shown that W is a countable union of closed sets $\{A_n\}$ of X. Using normality of X, for each n, choose a continuous function $f_n : X \to [0, 1]$ such that $f_n(A_n) = \{1\}$ and $f_n(X \setminus W) = \{0\}$. Let

$$f(x) = \Sigma f_n(x)/2^n.$$

Since the series converges uniformly, so the limit function f is continuous. Clearly $f(x) > 0 \ \forall x \in W$ and $f(X \setminus W) = \{0\}$.

Step 2. We can write $\mathscr{B} = \bigcup_n \mathscr{B}_n$ where each collection $\mathscr{B}_n$ is locally finite. For each $n \in N$ and $B \in \mathscr{B}_n$, choose a continuous function

$$f_{n,B} : X \to [0, 1/n]$$

such that $f_{n,B}(x) > 0$ for $x \in B$ and $f_{n,B}(X \setminus B) = \{0\}$. Now given any point $x_0 \in X$ and open set U containing x_0, ∃ a basis element B such that $x_0 \in B \subset U$. Then $B \in \mathscr{B}_n$ for some n and hence $f_{n,B}(x_0) > 0$ and $f_{n,B}(X \setminus U) = \{0\}$. In other words the collection of functions $\{f_{n,B}\}$ separates points from closed sets. Since X is T_1, so it also separates points.

Let J be the subset of $N \times \mathscr{B}$ consisting of all pairs (n, B) such that $B \in \mathscr{B}_n$. Define

$$F : X \to [0, 1]^J$$

by the equation

$$F(x) = (f_{n,B}(x))_{(n,B) \in J}.$$

By Imbedding theorem F is an imbedding of X into $[0, 1]^J$ with the product topology.

Step 3. It should be noted that $[0, 1]^J$ with the product topology is not always metrizable (if J is uncountable). So instead of taking the product topology, we take the uniform topology induced by the uniform metric $\bar{\rho}$ on $[0, 1]^J$.

Since the uniform topology is finer than the product topology, so F is still an open map. Evidently $F : X \to F(X)$ is bijective. We have only to show that F is continuous. Let $x_0 \in X$ and $\in > 0$ be given.

First let $n \in N$ be fixed. Since $\mathscr{B}_n$ is locally finite, ∃ a neighbourhood U_n of x_0 which meets only a finite number of members of $\mathscr{B}_n$, say, $B_1, \ldots, B_k$. Then

$$f_{n,B}(U_n) = \{0\}, \quad \forall B \in \mathcal{B}_n \setminus \{B_1 ..., B_k\}$$

Again by continuity of f_{n,B_i}, $\exists$ a neighbourhood V'_i of x_0 such that

$$y \in V'_i \Rightarrow |f_{n,B_i}(y) - f_{n,B_i}(x_0)| < \in/4 \text{ for } i = 1 \text{ to } k.$$

Let $V_n = V'_1 \cap .. \cap V'_k \cap U_n$. Then V_n is an open neighbourhood of x_0 and for any $x, y \in V_n$, any $i = 1$ to k.

$$\left|f_{n,B_i}(x) - f_{n,B_i}(y)\right| < \frac{\in}{2}$$

as V_n is a neighbourhood of x_0 on which all but finite functions, $f_{n,B}$ vanishes identically and the remaining functions $f_{n,B}$ vary atmost by $\frac{\in}{2}$.

Now choose $M \in N$ such that $1/M < \in/2$. For each of the (+)ve integers 1, 2 .., M, choose open neighbourhoods $V_1,.., V_M$ of x_0 having the above property. Let

$$W = V_1 \cap V_2 \cap .... \cap V_M.$$

Let $x \in W$. If $n \leq M$, then

$$|f_{n,B}(x) - f_{n,B}(x_0)| \leq \frac{\in}{2}$$

because each $f_{n,B}$ either vanishes identically or varies by at most $\frac{\in}{2}$ on W.

If $n > M$, then

$$|f_{n,B}(x) - f_{n,B}(x_0)| \leq \frac{1}{n} < \frac{\in}{2}$$

because $f_{n,B}$ maps X into $\left[0, \frac{1}{n}\right]$. Therefore

$$\bar{\rho}(F(x), F(x_0) \leq \frac{\in}{2} < \in.$$

Thus we have an open neighbourhood W of x_0 such that

$$x \in W \Rightarrow \bar{\rho}(F(x), F(x_0)) < \in.$$

This proves that the function F is continuous. This completes the proof of the sufficiency part.

Proof (necessity) :

Step 1. Let X be a metrizable space. First we prove that any open cover $\mathscr{V}$ of X has a σ-locally finite refinement $\mathscr{D}$ covering X.

Let d be a metric on X that induces the topology of X. Since the collection $\mathscr{V}$ is a poset (w.r.t inclusion) it can be well-ordered by well ordering theorem. Let '<' be the well ordering in $\mathscr{V}$. Let $n \in N$ be fixed. For $U \in \mathscr{V}$, define

$$S_n(U) = \left\{x \,;\, B\left(x, \frac{1}{n}\right) \subset U\right\}$$

Then we define

$$S'_n(U) = S_n(U) \setminus \bigcup_{V<U,\, V \in \mathscr{V}} V$$

We shall show that $\{S'_n(U) : U \in \mathscr{V}\}$ consists of pairwise disjoint sets. For this let $V, W \in \mathscr{V}, V \neq W$. Without any loss of generality we may assume that $V < W$. Now $x \in S'_n(V)$ $\Rightarrow x \in S_n(V)$. Again $y \in S'_n(W)$ implies by definition, $y \notin V$ ($\because V < W$). Clearly $x \in S_n(V), y \notin V$ $\Rightarrow d(x, y) \geq \frac{1}{n}$. Thus $S'_n(V) \cap S'_n(W) = \phi$ and

$$x \in S'_n(V) \text{ and } y \in S'_n(V) \Rightarrow d(x, y) \geq \frac{1}{n}.$$

Now let us define

$$E_n(U) = \cup\{B(x, 1/3n) : x \in S'_n(U)\}.$$

Evidently $E_n(U)$ is open for each $U \in \mathscr{V}$. For $V, W \in \mathscr{V}, V \neq W$, we assert that $E_n(V) \cap E_n(W) = \phi$. For if not, then $\exists$ *a* $z \in E_n(V) \cap E_n(W) \Rightarrow \exists x \in S'_n(V)$ and $y \in S'_n(W)$ such that $z \in B\left(x, \frac{1}{3n}\right)$ and $z \in B\left(y, \frac{1}{3n}\right) \Rightarrow d(x, y) \leq d(x, z) + d(z, y) < \frac{2}{3n} < \frac{1}{n}$, a contradiction.* Further it is easy to show that $E_n(U) \subset U$, $\forall\, U \in \mathscr{V}$. (* Actually it is easy to see that

$$x \in E_n(V),\ y \in E_n(W) \Rightarrow d(x, y) \geq \frac{1}{3n}.)$$

Now let us define

$$\mathscr{D}_n = \{E_n(U) : U \in \mathscr{V}\} \text{ for } n \in N.$$

Evidently $\mathscr{D}_n$ is a refinement of $\mathscr{V}$. Also for each $x \in X$, $B\left(x, \frac{1}{6n}\right)$ is an open neighbourhood of x which can intersect at most one member of $\mathscr{D}_n$. Thus $\mathscr{D}_n$ is locally finite.

Finally let $\mathscr{D} = \bigcup_{n \in N} \mathscr{D}_n$. Then $\mathscr{D}$ is a σ-locally finite refinement of $\mathscr{V}$. We just have to show that $\mathscr{D}$ covers X.

Let $x \in X$. Choose $U \in \mathscr{V}$ to be the first element that contains x (w.r.t. the well-ordering). Since U is open, choose $n \in N$ such that $B\left(x, \frac{1}{n}\right) \subset U$. Then $x \in S_n(U)$. Since there is no $V \in \mathscr{V}$ with $x \in V$, $V < U$. So clearly $x \in S'_n(U) \subset E_n(U)$. This completes the proof of our assertion.

Step 2. As X is metrizable, it is evidently regular and T_1. We have to show that X has a σ-locally finite basis. For this we note that for any $m \in N$,

$$\{B(x, \frac{1}{m}) : x \in X\} = \vartheta_m \text{ (say)}$$

is an open cover of X. By step 1, ϑ_m has a σ-locally finite refinement $\mathscr{D}_m$ covering X. Note that every member of $\mathscr{D}_m$ has diameter at most $2/m$. Let $\mathscr{D} = \bigcup_{m \in N} \mathscr{D}_n$. Evidently $\mathscr{D}$ is also σ-locally finite. Further given any $x \in X$ and $\in > 0$, choose $m \in N$ so that $1/m < \epsilon/2$. As $\mathscr{D}_m$ covers X, we can choose $D \in \mathscr{D}_m$ such that $x \in D$. But as diam $(D) \leq 2/m$, so $D \subset B(x, \in)$. Hence $\mathscr{D}$ is a basis of X as $\bigcup_{m \in N} \vartheta_m$ is already a basis of X.

Theorem : (stone) Every metrizable space is paracompact.

The results follows from Lemma 2(paracompactness chapter) and the above Theorem.

Exercise : Let (X, d) be a metric space. Let $X \times X$ be endowed with the corresponding product topology. Then $d : X \times X \to R$ given by $(x, y) \to d(x, y)$ is a continuous function.

Solution : Let $\in > 0$ be given. Consider the open interval $(d(x, y) - \in, d(x, y) + \in)$. Choose the basic open set $B\left(x, \frac{\in}{2}\right) \times B\left(y, \frac{\in}{2}\right)$ containing (x, y) in the product topology in $X \times X$. Take $(x_1, y_1) \in B\left(x, \frac{\in}{2}\right) \times B\left(y, \frac{\in}{2}\right)$. Then $d(x, x_1) < \frac{\in}{2}$ and $d(y, y_1) < \frac{\in}{2}$.

Note that

$$d(x, y) \leq d(x, x_1) + d(x_1, y_1) + d(y_1, y)$$

i.e., $d(x, y) - d(x_1, y_1) \leq d(x, x_1) + d(y, y_1)$

i.e., $| d(x, y) - d(x_1, y_1)| \leq d(x, x_1) + d(y, y_1) < \in$

i.e., $d(x_1, y_1) \in (d(x, y) - \in, d(x, y) + \in)$.

This shows that d is continuous.

Exercise : Show that in a compact metrizable space X, every metric for X is a B-metric (bounded metric).

Solution : In order to prove the result we show that there exist points $a, b \in X$ such that $d(a, b) = \text{diam}(X)$ where d is the metric on X corresponding to the given topology. Let $X^* = X \times X$ with the product topology and let $f : X^* \to R$ be defined as before $f((x_1, x_2)) = d(x_1, x_2)$. We have already shown that f is a continuous mapping to R. Since X is compact, $X \times X = X^*$ is also compact. Then $f(X^*)$ being continuous image of a compact set is also a compact subset of R. Consequently $f(X^*)$ is closed and bounded in R. Let $C = \text{lub}\, f(X^*)$. Then $C \in f(X^*)$ and there exists $p \in X^*$ such that $f(p) = C$. Let $p = (a, b)$. Obviously $d(a, b) = \text{diam}(X)$ and the assertion follows immediately.

We will now show that the cartesian product of countably many metrizable spaces is also metrizable.

Theorem : Let $\{(X_n, d_n)\}_n$ be a countable family of metrizable spaces. Let diam $(X_n) \le M$ for all large n and diam $(X_n) \to 0$ as $n \to \infty$. Let us define $e(x, y) = \sup_n \{d_n(x_n, y_n)\}$.

Then τ_e (the topology corresponding to e) is the product topology of $\prod_n (X_n, \tau d_n)$.

Proof : Let for $n_0 \in N$, diam $(X_n) \le M \; \forall n \ge n_0$. Clearly e is well-defined. We only show the triangle inequality to prove that e is a metric. For $x = (x_n)_n$, $y = (y_n)_n$, $z = (z_n)_n \in \Pi Xn$,

$$e(x, z) = \sup_n d_n(x_n, z_n) \le \sup_n \{d_n(x_n, y_n) + d_n(y_n, z_n)\}$$

$$\le \sup_n d_n(x_n, y_n) + \sup_n d_n(y_n, z_n)$$

$$= e(x, y) + e(y, z).$$

To show that the product topology is given by the metric e, let $x = (x_n)_n \in \Pi X_n$ and

$$x = (x_n)_n \in S(x_1, \in_1) \times ... \times S(x_n, \in_n) \times \prod_{n+1}^{\infty} X_n = U(say).$$

Choose $\in = \min \{\in_1, \in_2, .., \in_n\}$. Then $\in > 0$ and $y \in B_e(x, \in) \Rightarrow e(x, y) < \in \Rightarrow$ $\sup_n d_n(x_n, y_n) < \in$

$\Rightarrow d_i(x_i, y_i) < \in_i$ for $1 \le i \le n$.

Hence $y \in U$ and so $B(x, \in) \subset U$. Thus one side is proved.

To prove the converse take a ball $B_e(x, \in)$. Since diam $(X_n) \to 0$, $\exists \, n_0 \in N$ with diam $(X_n) < \in/2 \;\; \forall n \ge n_0$. Let

$$U = B(x_1, \in/2) \times ... \times B(x_{n_0}, \in/2) \times \prod_{n_0+1}^{\infty} X_n$$

Take $y = (y_n)_n \in U$. Clearly $d_i(x_i, y_i) < \in/2 \;\; \forall \, 1 \le i \le n_0$.

Obviously then $y \in U \Rightarrow e(x, y) \le \in/2 \Rightarrow y \in B_e(x, \in)$.

This completes the proof.

Theorem : Let $\{X_n : n \in N\}$ be a family of metrizable spaces. Then $\prod_{n=1}^{\infty} X_n$ with product topology τ is metrizable.

Proof : Let d_n be the metric for X_n. We know that e_n for each X_n where $e_n(x, y) = \min\left\{\frac{1}{n}, d_n(x,y)\right\}$ for $(x, y) \in X_n \times X_n$ induces the same topology as d_n. Obviously diam $(X_n) \to 0$ as $n \to \infty$ and of course it is bounded uniformly. If $e(x, y) = \sup_n e_n(x_n, y_n)$ then $\tau_e = \tau$ and the result is proved.

Exercise : Let (X_1, d_1) and (X_2, d_2) be two metric spaces, $K \subset X_1$ and $f : K \to X_2$ be uniformly continuous. Then show that the oscillation $w(p)$ of f is zero at every $p \in X_1$.

Solution : For each $p \in K$, $w(p) = 0$ follows from continuity of f at p. If $p \in X_1 \setminus \bar{K}$ then we have

$$o \leq w(p) \leq \text{diam } (f(X_1 \setminus \bar{K}) \cap K) = 0$$

as $X_1 \setminus \bar{K}$ is an open neighbourhood of p.

Finally let $p \in \bar{K} \setminus K$. Let $\in > 0$ be given. By the uniform continuity of f, $\exists$ a $\delta > 0$ such that

$$d_1(r, q) < \delta \Rightarrow d_2(f(r), f(q)) < \in/2.$$

Choose $r, q \in B(p, \frac{\delta}{2}) \cap K$. Then $d_1(q, r) \leq d_1(p, q) + d_1(p, r) < \frac{\delta}{2} + \frac{\delta}{2} = \delta \Rightarrow d_2(f(q), f(r)) < \in/2$. Hence

$$\text{diam } (f(B(p, \delta/2) \cap K)) \leq \in/2 < \in.$$

Thus $0 \leq w(p) \leq \text{diam } (f(B(p, \delta/2) \cap K)) < \in$. Since this is true for any $\in > 0$, $w(p) = 0$.

3.2 Two important theorems

We first consider compactness in $C[a, b]$ the space of all real continuous functions on $[a, b]$ endowed with sup metric ρ. $C[a, b]$ itself is not compact as it is not bounded.

Definition : Let M be a class of real functions defined on $[a, b]$. M is said to be uniformly bounded if $\exists$ a $k > 0$ such that $|f(t)| \le k \quad \forall t \in [a, b], \quad \forall f \in M$. M is said to be equi-continuous if for every $\in > 0$, $\exists$ a $\delta > 0$ such that $|f(t_1) - f(t_2)| < \in$ if $|t_1 - t_2| < \delta \quad \forall f \in M$.

We now prove the following characterization.

Arzela-Ascoli's Thorem

A set $M \subset C[a, b]$ is relatively compact iff it is uniformly bounded and equi-continuous.

Proof : Suppose that M is relatively compact in $C[a, b]$. Then M is bounded. This is equivalent to saying that for $b_1(t) \in C[a, b]$, there is a $k > 0$ such that

$$\rho(x, b_1) = \sup_{a \le t \le b} |x(t) - b_1(t)| \le K \quad \forall x \in M.$$

So $\forall x \in M$,

$$\sup_{a \le t \le b} |x(t)| \le \sup_{a \le t \le b} |x(t) - b_1(t)| + \sup_{a \le t \le b} |b_1(t)| \le k + k' \text{ (say).}$$

This shows that M is uniformly bounded.

To prove equi-continuity, choose $\in > 0$ and construct a finite $\frac{\varepsilon}{3}$- net,

$A = \{x_1(t), x_2(t), \dots, x_k(t)\}$ for M ($\because$ M is totally bounded). The functions $x_i(t)$ are continuous and so uniformly continuous on $[a, b]$ $\forall i = 1, 2 \dots, n$. So $\exists$ $\delta_i > 0$ such that $|x_i(t_1) - x_i(t_2)| < \frac{\varepsilon}{3}$ when $|t_1 - t_2| < \delta_i$, $t_1, t_2 \in [a, b]$ for $i = 1, 2, .., k$. Choose $\delta = \min \{\delta_1, \dots, \delta_n\}$. Then for $i = 1, 2, .., k$ $|x_i(t_1) - x_i(t_2)| < \frac{\varepsilon}{3}$ when $|t_1 - t_2)| < \delta$, $t_1, t_2 \in [a, b]$. Let $x(t) \in M$. There exists a $x_i(t) \in A$ such that $\rho(x, x_i) < \frac{\varepsilon}{3}$.

Then if $|t_1 - t_2| < \delta$, $t_1, t_2 \in [a, b]$,

$$|x(t_1) - x(t_2)| \leq |x(t_1) - x_i(t_1)| + |x_i(t_1) - x_i(t_2)| + |x_i(t_2) - x(t_2)|$$

$$< \rho(x, x_i) + \frac{\varepsilon}{3} + \rho(x, x_i) < \varepsilon.$$

This is evidently true for any $x(t) \in M$ and so M is equi-continuous. Conversely suppose that $M \subset C[a, b]$ is uniformly bounded and equi-continuous. To show that M is relatively compact it is sufficient to prove that it is totally bounded. Suppose K is a (+)ve integer such that $|x(t)| \leq K \ \forall x \in M$ and $t \in [a, b]$. Let $\varepsilon > 0$ be given. Since M is equicontinuous, choose $\delta > 0$ such that $|x(t_1) - x(t_2)| < \frac{\varepsilon}{4} \ \forall x \in M$ when $|t_1 - t_2| < \delta$. Since $[a, b]$ in compact, it has a δ-net $t_1, \ldots, t_n$. Choose a (+)ve integer m such that $\frac{1}{m} < \frac{\varepsilon}{4}$ and divide $[-K, K]$ into 2km equal parts by the points

$y_0 = -K < y_1 < y_2 < \ldots < y_k = K$. where $K = 2Km$.

Consider those n-tuples $(y_{i_1}, y_{i_2}, \ldots, y_{i_n})$ of the numbers y_i, $i = 0, \ldots, k$ such that some $x \in M$ has the property that

$$\left|x(t_j) - y_{i_j}\right| < \frac{\varepsilon}{4}, j = 1, 2 \ldots, n$$

and choose one such $x \in M$ for each such n-tuple.

We shall show that the resulting finite subset E of M in an ε-net for M. Let $x \in M$; choose $y_{i_1}, y_{i_2}, \ldots, y_{i_n}$ so that $\left|x(t_j) - y_{i_j}\right| < \frac{\varepsilon}{4}, j = 1, 2 \ldots n$ and so there is a corresponding $e \in E$. Let $t \in [a, b]$ and choose j so that $|t - t_j| < \delta$. Then $|x(t) - e(t)| \leq |x(t) - x(t_j)| + |x(t_j) - y_{i_j}| + |y_{i_j} - e(t_j)| + |e(t_j) - e(t)| < \varepsilon$

Hence $\rho(x, e) = \sup\limits_{a \leq t \leq b} |x(t) - e(t)| < \varepsilon$

Note : The above theorem can be generalized for any compact metric spa· X in place of $[a, b]$.

We next refer to the Weierstrass approximation theorem and its generalization to a compact space by stone ; recall that Weierstrass approximation theorem states that $P[a, b]$, the space of all polynomials on $[a, b]$ is dense in $C[a, b]$.

Let X be a compact metric space and let $C(X)$ be the space of continuous real functions on X with the usual metric

$$\rho(f,g) = \sup_{x\in X}|f(x) - g(x)|.$$

We define algebraic operations in $C(X)$ as follows : If $f, g \in C(X)$ and a is real, then, for $x\in X$.

$$(f + g)(x) = f(x) + g(x)$$

$$(fg)(x) = f(x)\, g(x)$$

$$(af)(x) = af(x)$$

A set $A \subset C(X)$ is called an algebra if $f, g\in A$ and a real imply $f + g, fg, af\in A$. If A is an algebra then it is easy to show that $\bar{A}$ is also an algebra.

Stone-Weierstrass Theorem

Let A be a closed algebra in $C(X)$, X a compact metric space. Assume that $1\in A$ and A separates points (i.e. if $x, y\in X$, $x \neq y$, $\exists\, f\in A$ for which $f(x) \neq f(y)$). Then $A = C(X)$.

Proof : We first show that $f\in A \Rightarrow |f| \in A$. First suppose that $\sup\{|f(x)| : x\in X\} \leq 1$. Let $\varepsilon > 0$ and let $p(t) = a_0 + a_1 t + ... + a_n t^n$ be a polynomial such that $||(t)| - p(t)| < \varepsilon$ $\forall\, t\in[-1, 1]$.

Then $p(f) = a_0 + a_1 f + ... + a_n f^n \in A$($\because$ A is an algebra) and $||f(x)| - p(f(x))| < \varepsilon \ \forall\, x\in X$.

This shows that $|f|$ is a limit of A and so $|f| \in A$ as A is closed. For any $f\in A$ we can choose a constant $a(\neq 0)$ such that $|af(x)| \leq 1 \quad \forall\, x\in X$. Then as above we can show that $|af| \in A$ and so $|f| \in A$.

We next note that if $f, g\in A$ then $\min(f, g)$ and $\max(f, g)$ are in A. It follows readily from the facts that

$$\max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|$$

$$\min(f, g) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|$$

and $|f - g| \in A$ if $f, g \in A$ and A is an algebra.

Next let $f \in C(X)$. Let $x, y \in X$, $x \neq y$. Let g be the function which takes constant value $f(x)$ at all points. Then $g \in A$. Since A separates points, $\exists\, h \in A$ such that $h(x) \neq h(y)$. Without any loss of generality assume that $h(x) = 0$. There is a constant a such that the function f_{xy} given by $f_{xy} = g + ah$. satisfies $f_{xy}(x) = f(x)$ and $f_{xy}(y) = f(y)$ and clearly $f_{xy} \in A$. Let $\varepsilon > 0$. Since $(f_{xy} - f)(y) = f(y) - f(y) = 0 < \varepsilon$.

From the continuity of $f_{xy} - f$ we can find an open ball s_y such that $y \in s_y$ and $f_{xy}(z) < f(z) + \varepsilon \quad \forall z \in s_y$.

Since X is compact the open cover $\{S_y : y \in X\}$ has a finite subcover, say, $Sy_1, Sy_2, \ldots, Sy_n$. Let $f_x = \min(f_{xy_1}, \ldots, f_{xy_n})$. Then $f_x \in A$, $f_x(x) = f(x)$ and for every $z \in X$, $f_x(z) < f(z) + \varepsilon$.

Again using the same argument for each $x \in X$, choose an open ball T_x such that $f_x(z) > f(z) - \varepsilon \quad \forall z \in T_x$.

Since X is compact, a finite number of these balls $T_{x_1}, \ldots, T_{x_m}$ covers X. Let $F = \max\left\{f_{x_1}, \ldots, f_{x_m}\right\}$.

Then $F \in A$ and $\forall z \in X$, $|f(z) - F(z)| < \varepsilon$. This proves the theorem.

Group–A
(Short questions)

1. Is the discrete topology defined on a non-empty set X metrizable? If so explain with reasons.
2. Is the real number space endowed with the cofinite topology metrizable? Answer with reasons.

3. Show that a metrizable space is normal.

4. In a metric space (X, d) prove that $x \in \bar{A}$ iff $d(x, A) = 0$ where $x \in X$, $A \subset X$.

5. Let (X, d) be a metric space and $A \subset X$. If p is a limit point of M then show that A contains an infinite sequence of distinct points converging to p.

6. Let $X = N \cup \{b\}$ where N is the set of natural numbers and $b \notin N$. Define

$$d(x, y) = 1 \text{ if } x, y \in N, x \neq y$$

$$d(b, x) = d(x, b) = 1 + \frac{1}{x} \text{ if } x \in N$$

$$d(x, y) = 0 \text{ if } x = y$$

Show that d is a metric on X. Find dist $(N, \{b\})$.

7. Let $f : (X, d) \to (Y, e)$ be uniformly continuous.

If $A, B \subset X$ be such that $d(A, B) = 0$, show that $e(f(A), f(B)) = 0$.

8. Is the real number space endowed with lower limit topology metrizable. Answer with reasons.

Group–B
(Long questions)

1. Prove that metrization is invariant under homeomorphism.
2. Show that the derived set of a countably compact set in a metric space is countably compact.
3. For a pseudo-metric space (X, d) if $Y = \{\{\bar{x}\} : x \in X\}$, define $e(\{\bar{x}\}, \{\bar{y}\}) = d(x, y)$.

   First show that $\{\bar{x}\} = \{y : d(x, y) = 0\}$. Then prove that e is a metric on Y.
4. Let K be a subset of a metric space (X, d) and let $r > 0$. Define $S_r(K) = \{x \in X : d(x, y) < r$ for at least one point $y \in K\}$. Prove that $S_r(K)$ is open.
5. Let $\{x_n\}_n$ and $\{y_n\}_n$ be Cauchy sequences in a metric space (X, d). Define a relation '$\sim$' as follows : $\{x_n\}_n \sim \{y_n\}_n$ iff. $\lim_{n\to\infty} d(x_n, y_n) = 0$. Then '$\sim$' is an equivalence

relation and let X^* be the collection of equivalence classes. For $x^*, y^* \in X^*$, define $d^*(x^*, y^*) = \lim_{n\to\infty} d(x_n, y_n)$ where $\{x_n\}_n \in x^*$, $\{y_n\}_n \in y^*$. Prove that the limit exists and the limit does not depend on the members closen from the equivalence classes.

6. Prove that (X^*, d^*) (described in (5) above) is a metric space.

7. (a) If a separable space is also metrizable then prove that the space has a countable basis.

   (b) Show that any finite subspace of a metrizable space is always discrete.

8. Prove that a topological space (X, τ) is metrizable iff there is a homeomorphism of X onto a subspace of some metric space.

Unit–V ☐ Uniform spaces and proximity spaces

Introduction

We are already familiar with the notions of topological spaces and metric spaces. Due to the presence of a distance function, the topology induced by a metric is much stronger and also we can define notions like Cauchy condition, completeness, uniform continuity in metric spaces which cannot be defined in general topological spaces. The theory of uniformity was developed to bridge this gap and it is a tool which can be seen as a structure which is stronger than a topological space but weaker than a metric space. The theory of uniform spaces is somewhat analogas to the theory of metric spaces but can be applied to a large number of spaces; in particularl to those spaces, not necessarily satisfying the axiom of countability (i.e., which cannot be metrizable). We will see that every uniformity induces a topology on a set, whereas every metric or more generally, every family of pseudo-metrics induces a uniformity on a set. We will study the conditions under which a given topology can be induced by a uniformity (i.e., when the topology is uniformizable) and when a given uniformity can be induced by a metric (i.e., when the uniform space is metrizable). We will also study many more properties of these spaces. Finally we will study another related structure, called proximity structure.

5.1. Basic definitions and properties

Let X be a nonempty set. A nonempty subsat U of $X \times X$ is called a relation on X. If U is a relation on X, its inverse relation U^{-1} is defined by $U^{-1} = \{(x, y) : (y, x) \in U\}$. Clearly $(U^{-1})^{-1} = U$. If $U^{-1} = U$, the relation U is said to be symmetric. If U and V are two relation on X, then their composition is defined by,

$UoV = \{(x, y) : (x, z) \in V$ and $(z, y) \in U$ for some z in $X\}$. It is easy to verify that for any three relations U, V, W on X, $(UoV)o\, W = Uo\,(VoW)$ and $(UoV)^{-1} = V^{-1}oU^{-1}$. We write $\Delta = \{(x, x) : x \in X\}$. For any relation U on X, $\Delta \circ U = U \circ \Delta = U$. So Δ is the identity relation on X.

Let A be any nonempty subset of X, U any relation on X and $x_o \in X$. We define $U[A] = \{y : (x, y) \in U$ for some x in $A\}$ and $U[x_0] = \{y : (x_0, y) \in U\}$.

If A is a subset of X and U and V are two relations on X, then we can verify that $(U \circ V)[A] = U[V[A]]$.

Uniformity and Uniform space.

Let X be a nonempty set. A nonempty family $\mathscr{U}$ of subsets of $X \times X$ is said to be a uniformity on X if the following hold :

(i) $\Delta \subset U$ for every $U \in \mathscr{U}$.

(ii) If $U \in \mathscr{U}$, then $U^{-1} \in \mathscr{U}$.

(iii) If $U \in \mathscr{U}$, there is a member V of $\mathscr{U}$ with $V \circ V \subset U$.

(iv) If U_1, U_2 are in $\mathscr{U}$, then $U_1 \cap U_2 \in \mathscr{U}$.

(v) If $W \subset X \times X$ and $U \subset W$ for some U in $\mathscr{U}$, then $W \in \mathscr{U}$.

The pair $(X, \mathscr{U})$ is called a uniform space.

Base and subbase of a uniformity.

Let $(X, \mathscr{U})$ be a uniform space. A nonempty subfamily $\mathscr{B}$ of $\mathscr{U}$ is said to be a base for the uniformity $\mathscr{U}$ if for every U in $\mathscr{U}$ there is a member V of $\mathscr{B}$ with $V \subset U$.

A nonempty subfamily $\mathscr{S}$ of $\mathscr{U}$ is said to be a subbase for the uniformity $\mathscr{U}$ if the family ($\mathscr{B}$ of all finite intersections of the members of $\mathscr{S}$) is base for the uniformity $\mathscr{U}$.

Theorem 1 : Let X be a nonempty set. A nonempty family β of subsets of $X \times X$ is base for some uniformity on X if the following hold.

(i) $\Delta \subset U$ for every $U \in \beta$.

(ii) If $U \in \beta$, there is a member V of β with $V \subset U^{-1}$.

(iii) If $U \in \beta$, there is a member V of β with $V o V \subset U$.

(iv) If U_1, U_2 are in β, then $U_1 \cap U_2$ contains a member of β.

Proof : First suppose that β is a base for the uniformity $\mathscr{U}$ on X. Then $\beta \subset \mathscr{U}$ and each member of $\mathscr{U}$ contains a member of β.

(i) Let $U \in \beta$. Then $U \in \mathscr{U}$ and so $\Delta \subset U$.

(ii) Let $U \in \beta$. Then $U \in \mathscr{U}$ and so $U^{-1} \in \mathscr{U}$.

So there is a member V of β with $V \subset U^{-1}$.

(iii) Let $U \in \beta$. Then $U \in \mathscr{U}$. So there is a member W of $\mathscr{U}$ with $W \, o \, W \subset U$. Again, W contains a member V of β. So $V \text{ o } V \subset W \text{ o } W \subset U$.

(iv) Let U_1, U_2 belong to β. Then U_1, U_2 are in $\mathscr{U}$ and so $U_1 \cap U_2 \in \mathscr{U}$. Hence there is a member V of β with $V \subset U_1 \cap U_2$.

Thus the conditions are necessary.

Next, let β be a nonempty family of subsets of $X \times X$ satisfying the given conditions. Denote by $\mathscr{U}$ the family of all those subsets $\cup$ of $X \times X$ such that $U \in \mathscr{U}$, iff $V \subset U$ for some V in β. Clearly $\beta \subset \mathscr{U}$.

(a) Let $U \in \mathscr{U}$. Then there is a member V of β with $V \subset U$. Since $\Delta \subset V$, $\Delta \subset U$.

(b) Let $U \in \mathscr{U}$. There is a member V of β with $V \subset U$. Also, there is a member W of β with $W \subset V^{-1}$. Since $V^{-1} \subset U^{-1}$, $W \subset U^{-1}$. So $U^{-1} \in \mathscr{U}$.

(c) Let $U \in \mathscr{U}$. Then $V \subset U$ for some V in β. So there is a member W of β such that $WoW \subset V$. Since $\beta \subset \mathscr{U}$, $W \in \mathscr{U}$. So, $WoW \subset U$.

(d) Let U_1, U_2 be two members of $\mathscr{U}$. Then there are two members V_1, V_2 in β with $V_1 \subset U_1$ and $V_2 \subset U_2$. By condition (iv), there is a member V of β with $V \subset V_1 \cap V_2$. So $V \subset U_1 \cap U_2$. This gives that $U_1 \cap U_2 \in \mathscr{U}$.

(e) Let W be a subset of $X \times X$ such that $U \subset W$ for some U in $\mathscr{U}$. There is a member V in β with $V \subset U$. So $V \subset W$ which gives that $W \in \mathscr{U}$.

Hence, $\mathscr{U}$ is a uniformity on X. From the construction of $\mathscr{U}$ it is obvious that β is a base for the uniformity $\mathscr{U}$.

Exercise : Let X be a nonempty set. A nonempty family s of subsets of $X \times X$ is a subbase for some uniformity on X if the following hold.

(i) $\Delta \subset U$ for every, $U \in s$.

(ii) If $U \in s$, then there exist finitely many members $V_1, ..., V_n$ (say) in s such that U^{-1} contains $V_1 \cap ... \cap V_n$.

(iii) If $U \in s$, then there exist finitely many members $V_1, ..., V_m$ (say) in s with $VoV \subset U$. Where $V = V_1 \cap ..... \cap V_n$.

Solution : Let the family s of subsets of $X \times X$ satisfy the given conditions.

Denote by β the collection of all finite intersections of the members of s. Clearly $s \subset \beta$.

(a) Let $U \in \beta$. Then $U = \cap_{i=1}^{n} U_i$, where $U_i \in s$. Since $\Delta \subset U_i$ for each i, $\Delta \subset \cap_{i=1}^{n} U_i = U$.

(b) Let $U \in \beta$. Then $U = \cap_{i=1}^{n} U_i$, where $U_i \in s$. We have $U^{-1} = \cap_{i=1}^{n} = U_i^{-1}$. For each i, there is a member V_i in s with $V_i \subset U_i^{-1}$. Write $V = \cap_{i=1}^{n} V_i$. Then $V \in \beta$. We have $V \subset \cap_{i=1}^{n} U_i^{-1} = U^{-1}$.

(c) Let $U \in \beta$. Then $U = \cap_{i=1}^{n} U_i$, where $U_i \in s$. For each i, there is a member W_i of s with $W_i o W_i \subset U_i$. Let $W = \cap_{i=1}^{n} W_i$. Then $W \in \beta$ and $WoW \subset \cap_{i=1}^{n} (W_i o W_i) \subset \cap_{i=1}^{n} U_i = U$.

(d) Let U and V be any two members of β. Then $U = \cap_{i=1}^{m} U_i$ and $V = \cap_{i=1}^{n} V_i$, where $U_i, V_i \in s$.

We have $U \cap V = U_1 \cap U_2 \cap .. \cap U_m \cap V_1 \cap V_2 \cap .. \cap V_n$. This gives that $U \cap V \in \beta$.

Thus β is a base for some uniformity $\mathscr{U}$ on X. From the construction of β it is clear that s is a subbase for the uniformity $\mathscr{U}$.

Theorem 3. Let $(X, \mathscr{U})$ be a uniform space and let τ denote the family consisting of the void set ϕ and all those subsets S of X such that if $x \in S$, then $U[x] \subset S$ for some U in $\mathscr{U}$. Then τ is a topology on X.

Let $x_0 \in X$ and $\mathscr{U} = \{U[x_0] : U \in \mathscr{U}\}$. Then $\mathscr{U}$ is a neighbourhood base at x_0.

Proof : Let $x \in X$ and $U \in \mathscr{U}$. Then $U[x] \subset X$ which gives that $X \in \tau$. Let S_1 and S_2 be any two members of τ. Write $S = S_1 \cap S_2$. If $S = \phi$, then $S \in \tau$. Suppose that $S \neq \phi$. Let $x \in S$.

Then $x \in S_1$ and $x \in S_2$. So there are members U_1, U_2 in $\mathscr{U}$ such that $U_1[x] \subset S_1$ and $U_2[x] \subset S_2$. Write $U = U_1 \cap U_2$. Then $U \in \mathscr{U}$ and $U[x] \subset U_1[x] \subset S_1$. $U[x] \subset U_2[x] \subset S_2$. So $U[x] \subset S_1 \cap S_2$. This gives that $S \in \tau$.

Let $\mathscr{F} = \{S_\alpha : \alpha \in \Delta\}$ be a nonempty subfamily of τ and let $S = \cup\{S_\alpha : \alpha \in \Delta\}$.

Let $x \in S$. Then $x \in S_\alpha$ for some α in Δ. There is a member U of $\mathcal{U}$ with $U[x] \subset S_\alpha$. Since $S_\alpha \subset S$, $U[x] \subset S$. So $S \in \tau$.

Therefore τ is a topology on X.

For the second part we proceed as follows. Let $x_0 \in X$ and U be a member of $\mathcal{U}$. Consider the set $U[x_0]$.

Let $A = \{x : V[x] \subset U[x_0]$ for some V in $\mathcal{U}\}$. Taking $V = U$, we have $V[x_0] \subset U[x_0]$ which gives that $x_0 \in A$. Also from definition we get $A \subset U[x_0]$. We now show that A is open.

Let $x \in A$. Then there is a member V in $\mathcal{U}$ with $V[x] \subset U[x_0]$. Choose a member W in $\mathcal{U}$ with $WoW \subset V$.

Take any $y \in W[x]$. Then $(x, y) \in W$. If $z \in W[y]$, then $(y, z) \in W$. So $(x, z) \in WoW \subset V$. This gives that $z \in V[x] \Rightarrow W[y] \subset U[x] \Rightarrow y \in A$.

Thus $W[x] \subset A$. So A is open.

Since $x_0 \in A \subset U[x_0]$, it follows that $U[x_0]$ is a neighbourhood of x_0.

Let W be any neighbourhood of x_0. Then there is an open set G with $x_0 \in G \subset W$. So there is a member U of $\mathcal{U}$ with $U[x_0] \subset G \subset W$.

This gives that the family.

$\mathcal{V} = \{U[x_0] : U \in \mathcal{U}\}$ is a neighbourhood base at x_0.

Note : We say that the topology τ on X in Theorem 3, generated by the uniformity $\mathcal{U}$, is the uniform topology.

Definition : A topological space (X, τ) is said to be uniformisable if there is a uniformity $\mathcal{U}$ on X such that the topology generated by the uniformity $\mathcal{U}$ is identical with the topology τ.

Example 1 : Let (X, d) be a pseudometric space. For positive number r, let

$W_r = \{(x, y) : x, y \in X$ and $d(x, y) < r\}$. and $\beta = \{W_r : r > 0\}$.

We verify that β is a base for some uniformity on X.

(i) Since $d(x, x) = 0$ for all $x \in X$, it follows that $\Delta \subset W_r$ for every $r > 0$.

(ii) Since $d(y, x) = d(x, y)$ for all x, y in X, we get $W_r^{-1} = W_r$ for $r > 0$. So $W_r^{-1} \in \beta$.

(iii) Let r be any positive number and let $p = \frac{1}{2}r$. Let $(x, y) \in W_p \circ W_p$. Then (z, z), $(x, y) \in W_p$ for some z in X. We have

$d(x, y) \leq d(x, z) + d(z, y) < 2p = r.$

This gives that $(x, y) \in W_r$ and so

$$W_p \circ W_p \subset W_r.$$

(iv) Let $Wr_1, Wr_2 \in \beta$. Let $r = \min \{r_1, r_2\}$. If $(x, y) \in W_r$, then $d(x, y) < r \leq r_i$ $(i = 1, 2)$ which gives that $W_r \subset W_{r_1}$ and $W_r \subset W_{r_2}$. So, $W_r \subset W_{r_1} \cap W_{r_2}$.

Hence β is a base for some uniformity $\mathscr{U}$ on X. A subset U of $X \times X$ belongs to $\mathscr{U}$ if $W_r \subset U$ for some $r > 0$.

We now show that the topology τ_1 generated by the pseudometric d is identical with the topology τ_2 generated by the uniformity $\mathscr{U}$.

Let $G \in \tau_1$ and $x \in G$. Then there is a positive number r such that

$$S(x, r ; d) \subset G,$$

where $S(x, r ; d) = \{y : y \in X \text{ and } d(x, y) < r\}$

$= W_r [x].$

Thus $W_r [x] \subset G$. Since $W_r \in \mathscr{U}$, $G \in \tau_2$.

Again, let $G \in \tau_2$ and $x \in G$. Then there is a member U in $\mathscr{U}$ with $U[x] \subset G$. Since β is a base for $\mathscr{U}$, there is a positive number r, such that $W_r \subset U$. So $W_r [x] \subset U[x] \subset G$.

Since $W_r [x] = S(x, r ; d)$, $S(x, r ; d) \subset G$. This gives that $G \in \tau_1$. Hence $\tau_1 = \tau_2$. Therefore the pseudometric space (X, d) is uniformisable.

5.2. Uniformizability and metrizability

Example 2. Let X be a nonempty set and ρ be a family of pseudometrics on X. For $d \in \theta$ and $r > 0$.

Let $W_{(d, r)} = (x, y) : x, y \in X$ and $d(x, y) < r\}$, and

$$s = \{W_{(d, r)} : d \in P \text{ and } r > 0\}.$$

Let $d \in \rho$ and $r > 0$. Since $d(x, x) = 0$ for all $x \in X$, we have $\Delta \subset W_{(d, r)}$. Again, since $d(y, x) = d(x, y)$ for all x, y in X, $W_{(d,r)}^{-1} = W_{(d,r)}$.

Also $W_{\left(d,\frac{1}{2}r\right)} o W_{\left(d,\frac{1}{2}r\right)} \subset W_{(d, r)}$.

Therefore s is a subbase for some uniformity u on X. Let β denote the family of all finite intersections of the members of s. Then β is a base for the uniformity u.

Let τ_1 denote the topology generated by the family ρ of pseudo-metrics and τ_2 the topology generated by the uniformity u.

Let $G \in \tau_1$ and $x \in G$. Then there is a set of the form

$B = \cap_{i=1}^{n} S(x, r_i, d_i)$, where $d_i \in \rho$ and $r_2 > 0$, such that $x \in B \subset G$.

Since $S(x, r_i, d_i) = W_{(d_i,r_i)}[x]$, we have $B = \cap_{i=1}^{n} S(x, r_i, d_i) = W[x]$, where $W = \cap_{i=1}^{n} W_{(d_i,r_i)} \in u$.

This gives that $G \in \tau_2$.

Again, let $G \in \tau_2$ and $x \in G$. Then there is a member U in u with $U[x] \subset G$. Since β is a base for u, there is a set of the form

$W = \cap_{i=1}^{n} W_{(d_i,r_i)}$ where $d_i \in \theta$ and $r_i > 0$ such that $W \subset U$.

So $W[x] \subset U[x] \subset G$.

Since $W[x] = \cap_{i=1}^{n} S(x, r_i, d_i)$, we have

$\cap_{i=1}^{n} S(x, r_i, d_i) \subset G$. So $G \in \tau_1$.

Hence $\tau_1 = \tau_2$.

Theorem 4 : Every completely regular space is uniformisable.

Proof : Let (X, τ) be a completely regular space. We first show that its topology τ can be generated by a family P of pseudometrics on X.

In fact, let us denote the family of all real valued continuous functions defined on

X by $C(X)$ and let $C^*(X)$ denote the subfamily of $C(X)$ consisting of bounded functions. For a finite number of functions $f_1, f_2, .., f_k \in C^*(X)$ define.

$$\rho_{f_1,f_2,..f_k}(x,y) = \max\{|f_1(x) - f_1(y)|, |f_2(x) - f_2(y)|, ..., |f_k(x) - f_k(y)|\}$$

for $x, y \in X$. It is easy to verify that $\rho_{f_1,f_2,..f_k}$ is a pseudo-metric on X. Let us consider the family ρ of all these pseudo-metrics $\rho_{f_1,f_2,..f_k}$ where $f_1, f_2, .., f_k \in C^*(X)$.

Observe that from the construction it follows that every $\rho : X \times X \to R$ is continuous where $\rho \in \rho$. Let τ_1 be the topology induced by ρ. Let $G \in \tau_1$ and $x \in G$. Then we have

$$\bigcap_{i=1}^{n} S(x, r_i, \rho_i) = \{y \in X : \rho_i(x,y) < r_i\} \subset G \text{ for some } r_i > 0 \text{ and } \rho_i \in \rho,\ i = 1, 2, .., n.$$

Take a fixed i. If ρ_i is generated by $f_1, f_2, . . , f_k$ (say) $\in C^*(X)$ then clearly

$$x \in \bigcap_{i=1}^{k} f_i^{-1}\ (f_i(x) - r_i, f_i(x) + r_i) \subset S(x, r_i, \rho_i).$$

As each f_i is continuous so $\bigcap_{i=1}^{k} f_i^{-1}(f_i(x) - r_i, f_i(x) + r_i) = v_i$ (say) $\in \tau$ and this shows that

$$x \in \bigcap_{i=1}^{n} v_i = v \text{ (say) } \subset G,$$

where $v \in \tau$. Hence $G \in \tau$.

On the other hand is $U \in \tau$ and $x \in U$, by complete regularity of X, $\exists$ a function $f \in C^*(X)$ such that $f(x) = 0$ and $f(y) = 1\ \forall y \in X \setminus U$. Then clearly $\rho_f \in P$ and

$$x \in S(x, \tfrac{1}{2}, \rho_f) \subset U.$$

This shows that $U \in \tau_1$. Therefore $\tau = \tau_1$.

For d in P and $r > 0$, let

$$W_{(d,r)} = \{(x,y) : x, y \in X \text{ and } d(x, y) < r\}$$

and $s = \{W_{(d,r)} : d \in p \text{ and } r > o\}$.

Let $d \in \theta$ and $r > 0$. Since $d(x, x) = 0$ for all $x \in X$, $\Delta \subset W_{(d, r)}$. Again, since $d(y, x) = d(x, y)$ for all x, y in X, we get $W^{-1}_{(d,r)} = W_{(d,r)}$. Also $W_{(d,\frac{1}{2}r)} o W_{(d,\frac{1}{2}r)} \subset W_{(d,r)}$.

Therefore s is a base for some uniformity u on X. Denote by β the family of all finite intersections of the members of s. Thus β is a base for the uniformity u. Let τ_0 denote the topology generated by the uniformity u.

Let $G \in \tau$ and $x \in G$. Then as before $d_i \in P$ and $r_i > 0$ there is a such that $x \in B = S(x, r_i, d_i) \subset G$.

Since $S(x, r_i, d_i) = W_{(d_i,r_i)}[x] \in u$,

Thus $W[x] \subset G$ which gives that $G \in \tau_0$. Again, let $G \in \tau_0$ and $x \in G$. Then there is a member U of u with $U[x] \subset G$. Since β is a base for u, there is a set of the form

$$W = \cap_{i=1}^{n} W_{(d_i,r_i)} \quad (d_i \in \theta \text{ and } r_i)$$

such that $W \subset U$. Thus $W[x] \subset U[x] \subset G$.

Since $W[x] = \cap_{i=1}^{n} S(x, r_i, d_i)$, it follows that $G \in \tau$. Hence $\tau = \tau_0$.

Therefore the space (X, τ) is uniformisable.

Definition : Let X be a nonempty set. A mapping $q : X \times X \rightarrow R$ is said to be a quasimetric on X if the following hold.

(i) $q(x, y) \geq 0$ and $q(x, x) = 0$.

(ii) $q(x, y) \leq q(x, z) + q(z, y)$

for x, y, z in X.

Theorem 5. (Metrization Lemma) :

Let X be a nonempty set and let $\{U_n\}_{n=0}^{\infty}$ be a sequence of subsets of $X \times X$ such that (i) $U_0 = X \times X$ (ii) $\Delta \subset U_n$ for each n and

(iii) $U_n \circ U_n \circ U_n \subset U_{n-1}$ $(n = 1, 2, 3, \ldots)$.

Then there exists a quasimetric q on X such that

(a) $U_n \subset \{(x, y) : q(x, y) < 2^{-n+2}\} \subset U_{n-1}$ $(n = 1, 2, 3, \ldots)$.

If each U_n is symmetric, q becomes a pseudometric.

Proof : It is easy to see that $U_n \subset U_{n-1}$ for $n = 1, 2, 3, \ldots$ We define a mapping $f : X \times X \to [0, \infty)$ as follows.

$$f(x, y) = 2^{-n+2}, \text{ if } (x, y) \in U_{n-1} \setminus U_n.$$
$$= 0, \text{ if } (x, y) \in U_n \text{ for all } n.$$

Now we define the mapping $q : X \times X \to [0, \infty]$ as follows : Let $(x, y) \in X \times X$. Then

$$q(x, y) = \inf \left\{ \sum_{i=1}^{k} f(x_{i-1}, x_i) \right\},$$

where the infimium is taken over all finite sequences $\{x_0, x_1, x_2, \ldots, x_k\} \subset X$ with $x_0 = x$, $x_k = y$.

It is obvious that $q(x, y) \geq 0$ and $q(x, x) = 0$.

Let x, y, z be three points of X. Choose any $\in > 0$. Thus there are finite sequences $\{x_0, x_1, x_2, \ldots, x_k\}$ and $\{z_0, z_1, z_2, \ldots, z_r\}$ in X with $x_0 = x$, $x_k = z$, $z_0 = z$ and $z_r = y$. Such that

$$\sum_{i=1}^{k} f(x_{i-1}, x_i) < q(x, z) + \frac{1}{2}\varepsilon$$

$$\sum_{i=1}^{r} f(z_{i-1}, z_i) < q(z, y) + \frac{1}{2}\varepsilon$$

Now, $\{x_0, x_1, x_2, \ldots, x_k, z_1, z_2, \ldots, z_r\}$ is a finite sequence in X with $x_0 = x$, $z_r = y$. So.

$$q(x, y) \leq \sum_{i=1}^{k} f(x_{i-1}, x_i) + \sum_{i=1}^{r} f(z_{i-1}, z_i).$$

Or, $q(x, y) < q(x, z) + q(z, y) + \varepsilon$.

This gives that

$$q(x, y) \leq q(x, z) + q(z, y).$$

Thus q is a quasimetric on X.

To prove the relation (a) we first prove the inequality

(b) $f(x_0, x_k) \leq 2\sum_{i=1}^{k} f(x_{i-1}, x_i)$

for any finite sequence $\{x_0, x_1, x_2, .., x_k\} \subset X$.

Clearly the inequality (b) holds for $k = 1$. Take any positive integer $m > 1$. Suppose that the inequality (b) holds for all positive integers $K < m$.

Let $S = \sum_{i=1}^{m} f(x_{i-1}, x_i)$ and $S > 0$.

We consider the following cases :

(i) $f(x_0, x_1) \leq \frac{1}{2}S$ and (ii) $f(x_0, x_1) > \frac{1}{2}S$.

Case (i) : Denote by k the largest positive integer such that

$\sum_{i=1}^{k} f(x_{i-1}, x_i) \leq \frac{1}{2}S$. Thus $k < m$

Clearly $\sum_{i=k+2}^{m} f(x_{i-1}, x_i) \leq \frac{1}{2}S$.

Also $f(x_k, x_{k+1}) \leq S$.

By induction hypothesis

$$f(x_0, x_k) \leq 2\sum_{i=1}^{k} f(x_{i-1}, x_i) \leq S$$

and $f(x_{k+1}, x_m) \leq 2\sum_{i=k+2}^{m} f(x_{i-1}, x_i) \leq S$.

Let n be the least positive integer such that $2^{-n+1} \leq S$. Then clearly (x_0, x_k), (x_k, x_{k+1}) and (x_{k+1}, x_m) all belong to U_n. Again since $U_n \circ U_n \circ U_n \subset U_{n-1}$, $(x_0, x_m) \in U_{n-1}$.

So $f(x_0, x_m) \le 2^{-n+2} \le 2S = 2\sum_{i=1}^{m} f(x_{i-1}, x_i)$.

Case (ii) : We have

$$f(x_0, x_1) \le S \text{ and } \sum_{i=2}^{m} f(x_{i-1}, x_i) < \frac{1}{2}S.$$

By induction hypothesis, $f(x_1, x_m) \le 2\sum_{i=2}^{m} f(x_{i-1}, x_i) < S$.

Let n denote the least positive integer such that $2^{-n+1} \le S$. Thus $(x_0, x_1), (x_1, x_m) \in U_n$.
So $(x_0, x_n) \in U_n \circ U_n = \Delta \circ U_n \circ U_n \subset U_{n-1}$.
This gives that

$$f(x_0, x_m) \le 2^{-n+2} \le 2S = 2\sum_{i=1}^{m} f(x_{i-1}, x_i)$$

Suppose that $S = 0$. Thus $f(x_{i-1}, x_i) = 0$ for i = 1, 2, .., m. Let n be any positive integer. Then $(x_0, x_1), (x_1, x_2) \in U_{n+1}$. So $(x_0, x_2) \in U_{n+1} \circ U_{n+1} \subset U_n$. This gives that $(x_0, x_2) \in \cap_{n=0}^{\infty} U_n = W$ (say).

Similarly $(x_0, x_3) \in W$, $(x_0, x_4) \in W$. At $(m-1)$ th step we get $(x_0, x_m) \in W$.

So $f(x_0, x_m) = 0$, and

$$f(x_0, x_m) = 2S = 2\sum_{i=1}^{m} f(x_{i-1}, x_m).$$

Thus in any case the inequality (b) also holds for $k = m$. Hence by the principle of finite induction (b) holds for every positive integer k.

Take any positive integer n and let $(x, y) \in U_n$.

Then $q(x, y) \le f(x, y) \le 2^{-n+1} < 2^{-n+2}$.

This gives that

(c) $U_n \subset \{(x, y) : q(x, y) < 2^{-n+2}\}$.

Now let $q(x, y) < 2^{-n+2}$. Thus there exists a finite sequence $\{x_0, x_1, x_2, ..., x_k\}$ in X with $x_0 = x$, $x_k = y$ such that

$$\sum_{i=1}^{k} f(x_{i-1}, x_i) < 2^{-n+2}.$$

By inequality (b) we have

$$f(x, y) = f(x_0, x_k) \leq 2\sum_{i=1}^{k} f(x_{i-1}, x_i) < 2^{-n+3}.$$

Since $f(x, y)$ takes values of the form 0, 2^{-p+2} (p = 1, 2, 3, ..) it follows that $f(x, y) \leq 2^{-n+2}$.

So $(x, y) \in U_{n-1}$ which gives that

(d) $\{(x, y) : q(x, y) < 2^{-n+2}\} \subset U_{n-1}$

Combining (c) and (d) we obtain (a).

If each U_n is symmetric, then $f(x, y) = f(y, x)$ for all x, y in X. This implies that $q(x, y) = q(y, x)$.

So q is a pseudometric.

Theorem 6. Every uniformity on a set X can be generated by a family of pseudometrics on X.

Proof : Let u be a uniformity on the set X. Let β be a base for the uniformity u such that each member of β is symmetric and is different from $X \times X$. For each V in β we choose a sequence $\{U_n(v)\}_{n=0}^{\infty}$ of symmetric sets in u such that

$$U_{n+1}^{(v)} \circ U_{n+1}^{(v)} \circ U_{n+1}^{(v)} \subset U_n^{(v)},$$

where $U_0^{(v)} = X \times X$ and $U_1^{(v)} = V$.

By metrization Lemma there exists a pseudometric d_v on X such that

(1) $U_n^{(v)} \subset \left\{(x,y) : d_v(x,y) < 2^{-n+2}\right\} \subset U_{n-1}^{(v)}$.

Let $P = \{d_v : V \in \beta\}$: Denote by γ the uniformity on X generated by the family P of pseudometrics on X. For $V \in \beta$ and $r > 0$, let

$$W_{(v, r)} = \{(x, y) : d_v (x, y) < r\}.$$

and $\beta_0 = \{W_{(v, r)} : V \in \beta \text{ and } r > 0\}$.

Thus β_0 is a subbase for the uniformity γ.

Let U be any member of u. Since β is a base for u, there is a member V in β with $V \subset U$. From (1) we have $W_{(v, 1)} \subset U_1^{(v)} = V$.

So $W_{(v, 1)} \subset U$. This gives that $U \in \gamma$.

Next let $W \in \gamma$. Then there is a set of the form,

$$\hat{W} = \cap_{i=1}^{k} W_{(v_i r_i)} \qquad (V_i \in \beta \text{ and } r_i > 0)$$

such that $\hat{W} \subset W$.

Choose positive integers $n_1, n_2, ..., n_k$ such that $2^{-n_i+2} < r_i (i = 1,2,...k)$. From (1) we have

$$U_{n_i}^{(v_i)} \subset \left\{(x,y) : dv_i(x,y) < 2^{-n_i+2}\right\} \subset W_{(v_i,r_i)}.$$

Write $U = \cap_{i=1}^{k} U_{n_i}^{(v_i)}$. Then $U \in u$ and

$$U \subset \cap_{i=1}^{k} W_{(v_i,r_i)} = \hat{W} \subset W.$$

This gives that $W \in u$.

Therefore $\mathscr{V} = u$. Which proves the theorem.

Theorem 7 : As in theorem 6, for every $V \in \mathscr{V}$ We can construct a sequence $\left\{U_n^v\right\}_{n=0}^{\infty}$ of symmetric sets in U such that

$U_0^v = X \times X,\ V_1^v = V$ and

$U_{n+1}^v \, o \, U_{n+1}^v \, o \, U_{n+1}^v \subset U_n^v \qquad \forall n \in N.$

By metrization Lemma there is a pseudometric d_v on X such that

$$U_n^v \subset \left\{(x,y) : d_v(x,y) < 2^{-n+2}\right\} \subset V_{n-1}^v.$$

clearly $\{(x, y) : d_v(x, y) < 1\} \subset U_1^v = v$.

We will first show that the pseudometric d_v thus constructed is a continuous function from $X \times X$ to R.

Take any point $(x_0, y_0) \in X \times X$ and choose $\in > 0$. We can find $n \in N$ so that $2^{-n+2} < \frac{\in}{2}$. Then taking $U_n^v = W$ (say) we have a $W \in \mathscr{V}$ such that

$$(x, y) \in W \Rightarrow d_v(x, y) < \frac{\in}{2}.$$

Consider the open neighborhood $W(x_0) \times W(y_0)$ of the point (x_0, y_0) in the product topology of $X \times X$. Clearly for $(x, y) \in W(x_0) \times W(y_0)$,

$$\left|d_v(x_0, y_0) - d_v(x, y)\right| \le d_v(x_0, x) + d_v(y, y_0) < \frac{\in}{2} + \frac{\in}{2} = \in.$$

This proves our assertion.

Finally to prove that (x, τ) is completely regular, choose $x_0 \in X$ and a closed set F, $x_0 \notin F$. Since τ is induced by the uniformity $\mathscr{V}$ So $\exists$ a $V \in \mathscr{V}$ such that $V[x_0] \cap F = \phi$. Define $f : X \to [0, 1]$ by $f(x) = \min\{1, d_v(x_0, x)\}$. Then f is continuous (by above assertion) and $f(x_0) = 0$ and $f(y) = 1 \, \forall y \in F$. This completes the proof.

Theorem 8 : A uniform space is pseudometrizable if its uniformity has a countable base.

Proof : Let (X, u) be a uniform space with a countable base $\{V_n\}_{n=0}^{\infty}$ for its uniformity u, where $V_0 = X \times X$.

First observe that $\beta' = \{V_n'\}_{n=0}^{\infty}$ where $V_n' = V_n \cap V_n^{-1}$ against forms a countable basis of $\mathscr{V}$ consisting of symmetric sets. Let $U_0 = V_0 = X \times X$. First choose $U'_1 \in V$ such that $U'_1 \circ U'_1 \subset U_0$ and then choose $U''_1 \in U$ such that $U_1'' \circ U_1'' \subset U'_1$. Then

$U''_1 \circ U''_1 \circ U''_1 \subset U''_1 \circ U''_1 \circ U''_1 \circ \Delta \subset U''_1 \circ U''_1 \circ U''_1 \circ U''_1 \subset U'_1 \circ U'_1 \subset U_0$.

Take $U'''_1 = U''_1 \cap V_1$ and choose a member U_1 (say) from B' such that $U_1 \subset U'''_1$. Then U_1 is symmetric,

$U_1 \subset V_1$ and $U_1 \circ U_1 \circ U_1 \subset U_0$.

For each positive integer n we proceed in this way.

Hence we obtain a sequence of symmetric sets $\{U_n\}_{n=0}^{\infty}$ in u which forms a base for u and possesses the properties.

(i) $U_0 = V_0$ (ii) $U_n \subset V_n$ and (iii) $U_n \circ U_n \circ U_n \subset U_{n-1}$.

By Metrization Lemma there exists a pseudometric d on X such that

(1) $U_n \subset \{(x, y) : d(x, y) < 2^{-n+2}\} \subset U_{n-1}$ $(n = 1, 2, 3, ...)$.

For any positive number r, let

$W_r = \{(x, y) : x, y \in X \text{ and } d(x, y) < r\}$

and $\beta = \{W_r : r > 0\}$.

Then β is a base for some uniformity $\mathscr{V}$ on X. We verify that $\mathscr{V} = u$.

Let $W \in u$. Since $\{U_n\}_{n=0}^{\infty}$ is a base for the uniformity u, $U_{n-1} \subset W$ for some positive integer n. Choose a positive number r with $r < 2^{-n+2}$. Then from (1) we have

$W_r \subset U_{n-1} \subset W$.

This gives that $W \in \mathscr{V}$.

Next, let $W \in \mathscr{V}$. Then $W_r \subset W$ for some $r > 0$. Choose a positive integer n with $2^{-n+2} < r$. Then from (1) we have $U_n \subset W_r \subset W$ which gives that $W \in u$.

Hence $\mathscr{V} = u$. Therefore the uniform space (X, u) is pseudometrizable.

Definition : Let (X, u) be a uniform space. A subset E of X is said to be totally bounded if for every U in u there are finite number of points x_1 x_2, ...x_n in X such that

$$E \subset U_{i=1}^{n} \ U[x_i].$$

Example 3 : Every compact subset of a uniform space is totally bounded.

Solution : Let (X, u) be a uniform space and let E be a compact subset of X. Take any U in u. For $x \in E$, $U[x]$ is a neighbourhood of x. So there is an open set G_x with $x \subset G_x \subset U[x]$. Let $\mathscr{F} = \{G_x : x \in E\}$. Then $\mathscr{F}$ is an open cover of the set E. Since E is compact, there are finite number of open sets G_{x_1}, G_{x_2}, ..., G_{x_n} in $\mathscr{F}$ such that

$$E \subset U_{i=1}^{n} G_{x_i}$$

Since $G_{x_i} \subset U[x_i]$. $(i = 1, 2, ..., n)$ we get

$$E \subset \cup_{i=1}^{n} U[x_i].$$

Hence E is totally bounded.

5.3 Cauchy nets and Cauchy filters : Completeness.

Let (X, u) be a uniform space. A net $\{S_n : n \in (D, \geq)\}$ in X is said to be a Cauchy net if for every U in u, there is an element n_0 in D such that

$$(X_m, x_n) \in U \text{ for all } m, n \text{ in } D \text{ with } m \geq n_0, n \geq n_0.$$

A filter $\mathscr{F}$ in X is said to be a Cauchy filter if for every U in u, there is a point p in X such that $U[p] \in \mathscr{F}$.

Completeness : A uniform space (X, u) is said to be complete if every Cauchy net in X is convergent.

Theorem 8 : A uniform space (X, u) is complete iff every Cauchy filter in X is convergent.

Proof : First suppose that the uniform space (X, u) is complete.

Take any Cauchy filter $\mathscr{F}$ in X. Let $\{s_A : A \in \mathscr{F}\}$ be a derived net of the filter $\mathscr{F}$.

Let U be any member of u. Choose a symmetric member V in u with $VoV \subset U$. Then there is a point x_0 in X such that $V[x_0] \in \mathscr{F}$.

Write $A_0 = V[x_0]$. Take any A, B in $\mathscr{F}$ with $A \subset A_0$, $B \subset A_0$. Then $s_A, s_B, \in A_0 = V[x_0]$.

So $(s_A, x_0) \in V$, and $(s_B, x_0) \in V$ which gives that $(s_A, s_B) \in V \circ V \subset U$. Thus $\{s_A, : A \in \mathscr{F}\}$ is a Cauchy net in X. Since (X, u) is complete, $\{s_A : A \in \mathscr{F}\}$ is convergent.

Let $p = \lim_{\mathscr{F}} s_A$. Let $A \in \mathscr{F}$. Choose any B in $\mathscr{F}$ with $B \subset A$. Then $s_B \in B \subset A$ which gives that $p \in \overline{A}$. So p is a cluster point of $\mathscr{F}$. Since $\mathscr{F}$ is a Cauchy filter, $\mathscr{F}$ converges to p. [see Ex.4 page 18].

Next suppose that every Cauchy filter in X converges. Let $\{s_n : n \in (D, \geq)\}$ be a Cauchy net in X. Denote by $\mathscr{F}$ the derived filter of the net $\{s_n : n \in D\}$. Take any U in u. Then there is an element n_0 in D such that $(x_m, x_n) \in U$ for all m, n in D with $m \geq n_0$, $n \geq n_0$. In particular

$$\left(x_m, x_{n_0}\right) \in U \text{ for all } n \text{ in } D \text{ with } n \geq n_0.$$

$$\text{Or, } x_n \in U\left[x_{n_0}\right] \text{ for all } n \text{ in } D \text{ with } n \geq n_0.$$

This gives that $U\left[x_{n_0}\right] \in \mathscr{F}$. Thus $\mathscr{F}$ is a Cauchy filter in X. By our hypothesis $\mathscr{F}$ converges to a point x_0 in X.

This completes the proof of the theorem.

Exercise : Let (X, u) be a uniform space and let $\mathscr{F}$ be a Cauchy filter in X. If x_0 is a cluster point of $\mathscr{F}$, then $\mathscr{F}$ converges to x_0.

Solution : Let U be any member of $\mathscr{U}$. Choose a symmetric member V of u with $VoVoV \subset U$. Since $\mathscr{F}$ is a Cauchy filter in X, there is a point p in X such that $V[p] \in \mathscr{F}$.

Again, since x_0 is a cluster point of $\mathscr{F}$, $x_0 \in \overline{V[p]}$. Then $V[x_0] \cap V[p] \neq \phi$. Let $z \in V[x_0] \cap V[p]$. Then $(z, x_0) \in V$ and $(z, p) \in V$. Let $u \in V[p]$.

Then $(u, p) \in V$. From above three we see that $(u, x_0) \in VoVoV \subset U$; so $u \in U[x_0]$ which gives that $V[p] \subset U[x_0]$. Thus $U[x_0] \in \mathscr{F}$.

Therefore $\mathscr{F}$ converges to x_0.

Theorem 9 : A uniform space is compact iff it is totally bounded and complete.

Proof : Let (X, u) be a uniform space.

First suppose that X is compact. Let U be any member of u. Take any point x in X. Then $U[x]$ is a neighbourhood of x. So there is an open set G_x such that $x \in G_x \subset U[x]$. Let

$$G = \{G_x : x \in X\}.$$

Then G is an open cover of X. Since X is compact there are finite number of open sets $G_{x_1}, G_{x_2}, \ldots G_{x_n}$ in G such that.

$$X = \cup_{i=1}^{n} G_{x_i}$$

Since $G_{x_i} \subset U[x_i]$, we have

$$X = \cup_{i=1}^{n} U[x_i].$$

So X is totally bounded.

Let $\mathscr{F}$ be a Cauchy filter in X. Since X is compact, $\mathscr{F}$ has a cluster point x_0 (say). So $\mathscr{F}$ converges to x_0. Hence the space X is complete.

Next, suppose that the space (X, u) is totally bounded and complete.

Let $\mathscr{F}$ be an ultrafilter in X. Take any member U in u. Since X is totally bounded, there are finite number of points $x_1, x_2, \ldots\ldots x_n$ in X such that

(1) $X = \cup_{i=1}^{n} U[x_i]$.

Since $\mathscr{F}$ is an ultrafilter and $X \in \mathscr{F}$, (1) implies that $U[x_i] \in \mathscr{F}$ for some i $(1 \le i \le n)$. So $\mathscr{F}$ is a Cauchy filter in X. Since X is complete, $\mathscr{F}$ is convergent. Hence the space X is compact.

This completes the proof of the theorem.

Definition : Let (X, u) and $(Y, \mathscr{V})$ be two uniform spaces and let $f : X \to Y$. The function f is said to be uniformly continuous if for every member V of $\mathscr{V}$, there is a member U in $\mathscr{V}$ such that

$(x', x'') \in U \Rightarrow (f(x'), f(x'')) \in V.$

Note : For any $f : X \to Y$, let us define $f_2 : X \times X \to Y \times Y$ as follows. For $(x', x'') \in X \times X, f_2 (x', x'') = (f(x^1), f(x''))$. Then the uniform continuity may be defined as follows : The function $f : X \to Y$ is said to be uniformly continuous if

$f_2^{-1} (V) \in u$ for every V in $\mathscr{V}$.

Theorem 10 : Let (X, u) and $(Y, \mathscr{V})$ be uniform spaces and let $f : X \to Y$ be continuous. If X is compact, then f is uniformly continuous.

Proof : Let V be any member of $\mathscr{V}$. Choose a symmetric member V_0 of $\mathscr{V}$ with $V_0 o V_0 \subset V$. Let $x \in X$. Since f is continuous, there is a symmetric member $W^{(x)}$ of u such that

(1) $u \in W^{(x)}[X] \Rightarrow f(u) \in v_0 [f(x)]$.

Choose a symmetric member $U^{(x)}$ in u with $U^{(x)}$ o $U^{(x)} \subset W^{(x)}$. Since $U^{(x)}[X]$ is a neighbourhood of x, there is an open set G_x with $x \in G_x \subset U^{(x)} [x]$.

Let $G = \{G_x : x \in X\}$.

Then G is an open cover of X. Since X is compact, we can select finite number of open sets $G_{x_1}, G_{x_2}, .., G_{x_n}$ from the family G such that

$$X = \cup_{i=1}^{n} G_{x_i}$$

Since $G_{x_i} \subset U^{(x_i)}[x_i]$, we have

(2) $X = U_{i=1}^{n} \; U^{(x_i)}[x_i]$

Let $U = \cap_{i=1}^{n} U^{(x_i)}$. Then $U \in u$.

Take any two points x', x'' in X with $(x', x'') \in U$. From (2) we see that $x' \in U^{(x_i)}[x_i]$ for some i $(1 \leq i \leq n)$. Thus $(x', x_i) \in U^{(x_i)}$; also $(x', x'') \in U \subset U^{(x_i)}$.

So $(x'', x_i) \in U^{(x_i)}$ o $U^{(x_i)} \subset W^{(x_i)}$, that is, $x'' \in W^{(x_i)} [x_i]$. Therefore by (1)

$(f(x''), f(x_i) \in V_0$ and $(f(x'), f(x_i)) \in V_0$.

So $(f(x'), f(x'') \in V_0$ o $V_0 \subset V$.

Hence f is uniformly continuous.

Theorem 11 : Let X be a nonempty set and let $\{(Y_a, \mathscr{V}_a) : a \in A\}$ be a family of uniform spaces and for each $a \in A$, $f_a : X \to Y_a$. Then there exists a smallest uniformity u on X relative to which the functions f_a are uniformly continuous.

Proof : We prove the theorem by the following steps.

(I) Let $s = \{(f_a)_2^{-1}(U_a) : U_a \in \mathscr{V}_a \text{ and } a \in A\}$.

We first verify that s is a subbase for some uniformity U on X. Let $\Delta = \{(x, x) : x \in X\}$ and $\Delta_a = \{(y_a, y_a) : y_a \in Y_a\}$.

(i) Let $U \in s$. Then $U = (f_a)_2^{-1}(U_a)$ for some $a \in A$ and $U_a \in \mathscr{V}_a$. Take any $x \in X$. Then $(f_a(x), f_a(x)) \in \Delta_a$ i.e. $(f_a)_2(x, x) \in \Delta_a \subset U_a$.

So $(x, x) \in (f_a)_2^{-1}(U_a)$ and hence $\Delta \subset U$.

(ii) Let $U \in s$. Then $U = (f_a)_2^{-1}(U_a)$ for some $a \in A$ and $U_a \in \mathscr{V}_a$. There is a member V_a in $\mathscr{V}_a$ with $V_a \circ V_a \subset U_a$.

Write $V = (f_a)_2^{-1}(V_a)$.

Let $(x, y) \in V \circ V$. Then there is an element z in X such that $(z, y) \in V$ and $(x, z) \in V$. This gives that $(f_a)_2(z, y) \in V_a$ and $(f_a)_2(x, z) \in V_a$.

i.e $(f_a(z), f_a(y)) \in V_a$ and $(f_a(x), f_a(z)) \in V_a$.

$\Rightarrow (f_a(x), f_a(y)) \in V_a \circ V_a \subset U_a$

i.e. $(f_a)_2(x, y) \in U_a$

$\Rightarrow (x, y) \in (f_a)_2^{-1}(U_a) = U.$

$\Rightarrow V \circ V \subset U$

(iii) Let $U \in s$. Then $U = (f_a)_2^{-1}(U_a)$ for some $a \in A$ and $U_a \in \mathscr{V}_a$. Since $U_a^{-1} \in \mathscr{V}_a$ we have

$$W = (f_a)_2^{-1}(U_a^{-1}) \in s.$$

Let $(x, y) \in W$. Then $(f_a)_2(x, y) \in U_a^{-1}$.

i.e. $(f_a(x), f_a(y)) \in U_a^{-1}$

$\Rightarrow (f_a(y), f_a(x)) \in U_a$

$\Rightarrow (y, x) \in (f_a)_2^{-1} (U_a) = U$

$\Rightarrow (x, y) \in U^{-1}$

$\Rightarrow W_* \subset U^{-1}$.

From (i), (ii) and (iii) we see that s is a subbase for some uniformity u on X.

(II) Let $a \in A$, consider the function f_a. Take any member V_a in $\mathscr{V}_a$. Then $U = (f_a)_2^{-1} (V_a) \in u$.

Let $(x, y) \in U$. Then $(f_a)_2 (x, y) \in V_a$ i.e. $(f_a(x), f_a(y)) \in V_a$.

Hence f_a is uniformly countinuous.

(III) Let $\tilde{u}$ be any uniformity on X relative to which the function f_a $(a \in A)$ are uniformly continuous.

Let $U \in u$. Then there is a set of the form

$W = \cap_{i=1}^{n} (f_{a_i})_2^{-1} (V_{a_i})$ $(a_i \in A, V_{a_i} \in \mathscr{V}_{a_i})$ such that $W \subset U$. Since the functions $fa_1, fa_2, \ldots f_{a_n}$ are uniformly countinous relative to the uniformity $\tilde{u}$, there are members $w_1, w_2, \ldots w_n$ in $\tilde{u}$ such that

$(x, y) \in W_i \Rightarrow (f_{a_i}(x), f_{a_i}(y)) \in V_{a_i}$ $(i = 1, 2, \ldots, n)$. Write $\tilde{W} = \cap_{i=1}^{n} W_i$. Let $(x, y) \in \tilde{W}$. Then $(x, y) \in W_i$ and so $(f_{a_i}(x), f_{a_i}(y)) \in V_{a_i}$

$$\Rightarrow (f_{a_i})_2 (x, y) \in V_{a_i} \Rightarrow (x, y) \in \bigcap_{i=1}^{n} (f_{a_i})_2^{-1} (V_{a_i}) = W \subset U.$$

So $\tilde{W} \subset U$. which gives that $U \in \tilde{u}$.

Hence $u \subset \tilde{u}$ and the proof is complete.

5.4. Proximity Spaces :

Definition : Let X be a nonempty set and let δ be a relation on the power set $P(X)$ of the set X. Suppose that δ satisfies the following axioms.

(1) For A, B in $P(X)$, $A \delta B \Rightarrow B \delta A$.

(2) Let A, B, C be in $P(X)$. Then $(A \cup B) \delta C$ iff $A \delta C$ or $B \delta C$.

(3) For A, B in $P(X)$, $A \delta B \Rightarrow A \neq \phi$ and $B \neq \phi$.

(4) Let A, B be in $P(X)$. Then $A \cap B \neq \phi \Rightarrow A \delta B$.

(5) Let A, B be in $P(X)$. If $A \bar{\delta} B$, there exists a subset E of X such that $A \bar{\delta} E$ and $\bar{E} \bar{\delta} B$, where $\bar{E}$ denotes the complement of E and $\bar{\delta}$ denotes the negation of δ.

Then δ is called a proximity on X and the pair (X, δ) is called a proximity space.

Example 1. Let (X, d) be a metric space. For two subsets A, B of X, let $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$.

Now define the relation δ on the power set $P(X)$ of the set X as follows : For A, B in $P(X)$, let

$$A \delta B \text{ hold iff } d(A, B) = 0$$

Then δ is a proximity on the set X.

Solution : (1) Since $d(y, x) = d(x, y)$ for all x, y in X, it follows that $d(B, A) = d(A, B)$ for A, B in $P(X)$. Let A, B be in $P(X)$ and $A \delta B$. Then $d(A, B) = 0$. So $d(B, A) = 0$ which gives that $B \delta A$.

(2) Let A, B, C be in $P(X)$. Suppose that $(A \cup B) \delta C$. Then $d(A \cup B, C) = 0$. Let $A \bar{\delta} C$. Then $d(A, C) = r > 0$. Choose any ε with $0 < \varepsilon < r$.

Since $d(A \cup B, C) = 0$, there is a point x in $A \cup B$ and a point Z in C such that

$$d(x, Z) < \varepsilon \; \ldots \quad \ldots \quad \ldots \quad (*)$$

If $x \in A$, then $d(x, Z) \geq d(A, C) = r > \varepsilon$. This contradicts (*). So $x \in B$ and $d(B, C) \leq d(x, Z) < \varepsilon$.

Since $\varepsilon > 0$ is arbitrary it follows that $d(B, C) = 0$. So $B \delta C$.

Next, let $A \delta C$. Then $d(A, C) = 0$. Choose any $\varepsilon > 0$. Then there is a point x in A and a point Z in C such that $d(x, Z) < \varepsilon$. Since $x \in A \cup B$, we have $d(A \cup B, C) \leq d(x, Z) < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $d(A \cup B, C) = 0$. This gives that $(A \cup B) \delta C$.

If $B \delta C$, as above we can show that $(A \cup B) \delta C$. Thus $(A \cup B) \delta C$ iff $A \delta C$ or $B \delta C$.

(3) Let A, B be in $P(X)$ and $A \delta B$. Choose any $\varepsilon > 0$. Then there is a point x in A and a point y in B with $d(x, y) < \varepsilon$. This gives that $A \neq \phi$ and $B \neq \phi$.

(4) Let A, B be in $P(X)$ and $A \cap B \neq \phi$. Take any $x \in A \cap B$. Then $x \in A$ and $x \in B$. Since $d(A, B) \leq d(x, x)$, we have $d(A, B) = 0$. So $A \, \delta \, B$.

(5) Let A, B be in $P(X)$ and let $A \bar{\delta} B$. Then

$$d(A, B) = r > 0 \quad \ldots \quad \ldots \quad \ldots \quad (**)$$

Let $E = \left\{ y : y \in X \text{ and } d(y, B) < \frac{1}{3} r \right\}$

Assume that $A \, \delta \, E$. Then $d(A, E) = 0$. So there is a point x in A and a point y in E with $d(x, y) < \frac{1}{3} r$. Since $d(y, B) < \frac{1}{3} r$, there is a point Z in B such that $d(y, Z) < \frac{1}{3} r$.

We have $d(x, Z) \leq d(x, y) + d(y, Z) < \frac{1}{3} r + \frac{1}{3} r = \frac{2}{3} r$. Since $d(A, B) \leq d(x, z)$, $d(A, B) \leq \frac{2}{3} r$.

This contradicts (**). Hence $A \bar{\delta} E$. Next, let $\tilde{E} \delta B$. Then $d(\tilde{E}, B) = 0$. So there is a point x in $\tilde{E}$ and a point y in B such that $d(x, y) < \frac{1}{3} r$. Since $d(x, B) \leq d(x, y)$, we have $d(x, B) < \frac{1}{3} r$ which gives that $x \in E$. This contradicts the fact that $x \in \tilde{E}$.

Hence $\tilde{E} \bar{\delta} B$.

Therefore δ is a proximity on X.

Example 2 : Let $(X, \mathcal{U})$ be a uniform space and let the relation δ be defined on $P(X)$ as follows : For A, B in $P(X)$, let $A \, \delta \, B$ if $(A \times B) \cap U \neq \phi$ for every U in $\mathcal{U}$. Then δ is a proximity on X.

Solution :

(i) Let A, B be in $P(X)$ and let $A \, \delta \, B$. Take any U in $\mathcal{U}$. Then $U^{-1} \in \mathcal{U}$ and so $(A \times B) \cap U^{-1} \neq \phi$. This gives that there is a point x in A and a point y in B such that $(x, y) \in U^{-1}$ or $(y, x) \in U$. So $(B \times A) \cap U \neq \phi$. Hence $B \delta A$.

(ii) Let A, B, C be in $\rho(X)$. Suppose that $(A \cup B) \, \delta \, C$. Assume that $A \bar{\delta} C$. Then there is a member V in $\mathcal{U}$ with $(A \times C) \cap V = \phi$. Take any U in $\mathcal{U}$. Write $W = U \cap V$. Then $W \in \mathcal{U}$. Since $(A \cup B) \delta C$, $[(A \cup B) \times C] \cap W \neq \phi$. This gives that there is a point x in $A \cup B$ and a point z in C such that $(x, Z) \in W$. If $x \in A$, then $(x, Z) \in A \times C$ and so $(x, Z) \in (A \times C) \cap W$

$\subset (A \times C) \cap V$ which condradicts the fact that $(A \times C) \cap V = \phi$. So $x \in B$. Hence $(x, Z) \in B \times C$; and $(x, Z) \in (B \times C) \cap U$. This gives that $(B \times C) \cap U \neq \phi$ and $B \ \delta \ C$.

Thus $(A \cup B) \delta C \Rightarrow$ either $A \delta C$ or $B \ \delta \ C$. Let $A \ \delta \ C$. Take any $U \in \mathscr{U}$. Thus $(A \times C) \cap U \neq \phi$. This gives that $[(A \cup B) \times C] \cap U \neq \phi$. So $(A \cup B) \delta C$. Similarly $B \ \delta \ C \Rightarrow (A \cup B) \delta C$.

(iii) Let A, B be in $P(X)$ and let $A \ \delta \ B$. Take any U is $\mathscr{U}$. Then $(A \times B) \cap U \neq \phi$. So there is a point x in A and a point y in B such that $(x, y) \in U$. This gives that $A \neq \phi$ and $B \neq \phi$.

(iv) Let A, $B \in \rho(X)$ and $A \cap B \neq \phi$. Take any point $x \in A \cup B$. Then $x \in A$ and $x \in B$. Let $U \in \mathscr{U}$. Since $(x, x) \in U$, we have $(A \times B) \cap U \neq \phi$. So $A \ \delta \ B$.

(v) Let A, $B \in P(X)$ and let $A \bar{\delta} B$. Thus there is a member U_0 in $\mathscr{U}$ such that

$(A \times B) \cap U_0 = \phi$ ... ... (1)

Choose a symmetric member V in $\mathscr{U}$ with $V_0 V \subset U_0$. Let

$E = \{y : y \in X$ and $(y, z) \in V$ for some Z in $B\}$. Assume that $A \delta E$. Then there is a point x in A and *a* point y in E with $(x, y) \in V$. Aslo from the definition of E, there is a point Z in B such that $(y, Z) \in V$. This gives that $(x, Z) \in V_0 V \subset U_0 \Rightarrow (A \times B) \cap U_0 \neq \phi$ which contradicts (1). Hence $A \bar{\delta} E$.

Again, assume that $\tilde{E} \delta B$. Then $(\tilde{E} \times B) \cap V \neq \phi$. So there is a point y in $\tilde{E}$ and a point Z in B such that $(y, Z) \in V$ which implies that $y \in E$. This contradicts the fact that $y \in \tilde{E}$. Hence $\tilde{E} \bar{\delta} B$.

Therefore δ is a proximity on X.

Lemma 1 : Let (X, δ) be a proximity space and let A, B, C, D be subsets of X.

(i) If $A \ \delta \ B$ and $A \subset C$ & $B \subset D$, then $C \ \delta \ D$.

(ii) If $x \in X$ and $A \delta x$ & $x \delta B$, then $A \delta B$.

(iii) If $A \bar{\delta} B$, then $\overline{A} \bar{\delta} B$, $A \bar{\delta} \overline{B}$ and $\overline{A} \bar{\delta} \overline{B}$, where $\overline{A} = \{y : y \in X$ and $y \ \delta \ A\}$.

(iv) $A \delta B$ if $\overline{A} \delta \overline{B}$, where $\overline{A}$ is defined as in (iii).

(v) If $A \bar{\delta} B$, then $\overline{B} \subset X \setminus A$ and $\overline{A} \subset X \setminus B$.

Proof : (i) Suppose that $A \ \delta \ B$ and $A \subset C$ & $B \subset D$. We have $C = A \cup (C \setminus A) =$

$A \cup E$, where $E = C \setminus A$. Since $A\delta B$, we get $(A \cup E)\ \delta\ B$ i.e., $C\delta B$. Again, since $D = B \cup (D \setminus B)$, we get $C\delta D$.

(ii) Suppose that there is a point x is X with

$A\delta x$ and $x\delta B$ ... ... (1)

Assume that $A\bar{\delta}B$. Then there is a subset E of X such that

$A\bar{\delta}E$ and $\tilde{E}\,\bar{\delta}B$ ... ... (2)

where $\tilde{E} = X \setminus E$.

If $x \in E$, then by (1) and (i) we have $A\delta E$ which contradicts (2). Again, if $x \in \tilde{E}$, then (1) and (i) imply that $\tilde{E}\delta B$ which also contradicts (2). Hence $A\delta B$.

(iii) Suppose that $A\bar{\delta}B$. Thus there is a subset E of X such that

$A\bar{\delta}E$ and $\tilde{E}\bar{\delta}B$ ... ... (3)

(a) Let $y \in \bar{B}$. Then $y \in B$. If $y \in \tilde{E}$, then $\tilde{E}\delta y$ and so by (ii) $\tilde{E}\delta B$ which contradicts (3). So y ∈ E ; this gives that $\bar{B} \subset E$. This with $A\bar{\delta}E$ and (i) imply that $A\bar{\delta}\bar{B}$.

Let $y \in \bar{A}$. Then $y\delta A$ and so $A\delta y$.

(b) If $y \in E$, then $y\delta E$ and by (ii) $A\delta E$ which contradicts (3). So $y \in \tilde{E}$; this gives that $\bar{A} \subset \tilde{E}$. If $\bar{A}\delta B$, then by (i) we get $\tilde{E}\delta B$ which contradicts (3). Hence $\bar{A}\bar{\delta}B$. Step (iiia) implies that $\overline{A\delta}\bar{B}$.

(iv) Suppose that $A\delta B$. Clearly $A \subset \bar{A}$ and $B \subset \bar{B}$. So by (i) $\bar{A}\delta\bar{B}$.

Next, suppose that $\bar{A}\delta\bar{B}$. ... ... (3a)

Assume that $A\bar{\delta}B$. Then by (iii) $\bar{A}\bar{\delta}\bar{B}$.

This contradicts (3a). Hence $A\delta B$.

(v) Suppose that $A\bar{\delta}B$. ... ... (4)

Let $y \in \bar{A}$. Then $y\delta A$ and so $A\delta y$. If $y \in B$, then by (ii) $A\delta B$ which contradicts (4). So $y \in X \setminus B$. Hence $\bar{A} \subset X \setminus B$.

Let $y \in \overline{B}$. Then $y\delta B$. If $y \in A$, then $A\delta y$ and by (ii) $A\delta B$ which contadits (4). So $y \in X\backslash A$. Hence $\overline{B} \subset X \backslash A$.

Theorem 1 : Let (X, δ) be a proximity space and for any subset A of X let

$$C(A) = \{y : y \in X \text{ and } y\delta A\}.$$

Then C is the Kuratowski closure operator on X.

Proof : (i) Let A be any subset of X. Take any x in A. There $\{x\} \cap A \neq \phi$; so $x \delta A$ which gives that $x \in C(A)$. Hence $A \subset C(A)$.

Clearly $C(\phi) = \phi$ and $C(X) = X$

(ii) Let A, B be two subsets of X and $A \subset B$. Take any $x \in C(A)$. Then $x \delta A$; so $x \delta B$. This gives that $x \in C(B)$.

Hence $C(A) \subset C(B)$.

(iii) Let A, B be any two subsets of X.

Take any $x \in C(A) \cup C(B)$.

Then $x \in C(A)$ or $x \in C(B)$

This gives that $x \delta A$ or $x \delta B$ and so

$x\delta(A \cup B). \Rightarrow x \in C(A \cup B)$.

Hence $C(A) \cup C(B) \subset C(A \cup B)$. ... ... (1)

Next, let $x \in C(A \cup B)$. Then $x \delta(A \cup B) \Rightarrow$

either $x\delta A$ or $x\delta B$. This gives that $x \in C(A)$ or $x \in C(B) \Rightarrow x \in C(A) \cup C(B)$.

So, $C(A \cup B) \subset C(A) \cup C(B)$ ... (2)

From (1) and (2) we have

$C(A \cup B) = C(A) \cup C(B)$.

(iv) Let A be any subset of X.

If $A = \phi$, then $C(A) = C(\phi) = \phi$ and so

$C(C(A)) = C(\phi) = \phi = A$

Suppose that $A \neq \phi$. By (i) we have

$$C(A) \subset C(C(A)). \quad \ldots \ldots \quad (3)$$

Let $y \in C(C(A))$. Then

$$y \delta C(A) \quad \ldots \quad \ldots (4)$$

Assume that $y \notin C(A)$. By the definition of $C(A)$ we get $y \bar{\delta} A$ and so $y \bar{\delta} C(A)$ which contradicts (4). So $y \in C(A)$ and

$$C(C(A)) = C(A).$$

Note : Let (X, δ) be a proximity space. For any subset A of X let $C(A) = \{y : y \in X$ and $y \delta A\}$. Then by theorem 1, C is the kuratowski closure operator on X. This closure operator C induces a topology $\tau_{(\delta)}$ on X. We say that the proximity δ induces the topology $\tau_{(\delta)}$ on X, and the topology $\tau_{(\delta)}$ is compatible with the proximity δ.

Theorem 2 : Let (X, τ) be a completely regular space. Then there exists a proximity δ on X compatible with the topology τ.

Proof : Since (X, τ) is completely regular, it is uniformisable. So there is a uniformity $\mathscr{U}$ on X such that the topology induced by the uniformity $\mathscr{U}$ is identical with the topology τ.

Now define the relation δ on $P(X)$, the power set of X, as follows :

For A, B in $P(X)$, $A \delta B$ if $(A \times B) \cap \mathscr{U} \neq \phi$ for every U in $\mathscr{U}$. Then δ is a proximity on X. The proximity δ induces a topology $\tau_{(\delta)}$ on X. Let A be any subset of X. Denote by $\bar{A}$ and $C(A)$ respectively the τ-closure and $\tau_{(\delta)}$-closure of the set A.

Let $x \in \bar{A}$. Take any $U \in \mathscr{U}$. Thus $U[x]$ is a neighd of x. So $A \cap U[x] \neq \phi$. This gives that there is a point y in A such that $y \in U[x]$ i.e. $(x, y) \in U$.

So, $(x, y) \in (\{x\} \times A) \cap U$

i.e. $(\{x\} \times A) \cap U \neq \phi \Rightarrow x \delta A$.

Hence $x \in C(A) \Rightarrow \bar{A} \subset C(A)$.

Next, let $x \in C(A)$.

Then $x\ \delta\ A \Rightarrow (\{x\} \times A) \cap U \neq \phi$ for any $U \in \mathscr{U}$

Let $U \in \mathscr{U}$. Then there is a point y in A such that $(x, y) \in U \Rightarrow y \in U[x]$

$\Rightarrow A \cap U[x] \neq \phi \Rightarrow x \in \overline{A}$

So, $C(A) \subset \overline{A}$

Thus $C(A) = \overline{A}$

This gives that $\tau_{(\delta)} = \tau$

Hence δ is complatible with τ.

Theorem 3 : Let (X, τ) be a T_4 space and let δ be a relation on the power set $P(X)$ defined as follows. For A, B in $\rho(X)$, $A\ \delta\ B$ iff $\overline{A} \cap \overline{B} \neq \phi$. Then δ is a proximity on X compatible with τ.

Proof : We first verify that δ is a proximity on X. (i) Let $A, B \in P(X)$ and $A\ \delta\ B$. Then $\overline{A} \cap \overline{B} \neq \phi$. Since $\overline{B} \cap \overline{A} = \overline{A} \cap \overline{B} \neq \phi$, we get $B\ \delta\ A$.

(ii) Let A, B, C be in $P(X)$.

Suppose that $(A \cup B)\ \delta\ c$.

Then $\overline{A \cup B} \cap \overline{C} \neq \phi$ i.e. $(\overline{A} \cup \overline{B}) \cap \overline{C} \neq \phi$

i.e. $(\overline{A} \cap \overline{C}) \cup (\overline{B} \cap \overline{C}) \neq \phi$

This gives that either $\overline{A} \cap \overline{C} \neq \phi$ or $\overline{B} \cap \overline{C} \neq \phi$.

so either $A\ \delta\ C$ or $B\ \delta\ C$.

Next, suppose that $A\ \delta\ C$. Then $\overline{A} \cap \overline{C} \neq \phi$.

This gives that $(\overline{A} \cap \overline{C}) \cup (\overline{B} \cap \overline{C}) \neq \phi$.

ie. $(\overline{A} \cup \overline{B}) \cap \overline{C} \neq \phi$

or, $(\overline{A \cup B}) \cap \overline{C} \neq \phi$

So, $(A \cup B)\ \delta\ C$.

If $B\ \delta\ C$, we can show that $(A \cup B)\ \delta\ C$.

(iii) Let $A, B \in P(X)$ and let $A\ \delta\ B$. Then $\overline{A} \cap \overline{B} \neq \phi$. If A = ϕ, then $\overline{A} = \phi$ and so $\overline{A} \cap \overline{B} = \phi$ which is a contradiction. Hence $A \neq \phi$. Similarly $B \neq \phi$.

(iv) Let $A, B \in P(X)$ and $A \cap B \neq \phi$. This gives that $\overline{A} \cap \overline{B} \neq \phi$ and so $A\ \delta\ B$.

(v) Let $A, B \in P(X)$ and $A \bar{\delta} B$. then $\overline{A} \cap \overline{B} = \phi$.

Since (X, τ) is a T_4 space it is normal. So there are open sets G_1, G_2 such that $\overline{A} \subset G_1$, $\overline{B} \subset G_2$ and $G_1 \cap G_2 = \phi$.

Write $E = X \backslash G_1$. Then E is a closed set. We have $\overline{A} \cap \overline{E} = \overline{A} \cap E = \phi$.

So $A \bar{\delta} E$.

Again, since $E \supset G_2 \supset \overline{B}$, $\tilde{E} \subset X | G_2 \subset X | \overline{B}$ and so $\text{cl}(\tilde{E}) \subset X | G_2 \subset X | \overline{B}$, where $\tilde{E} = X | E$.

This gives that cl $(\tilde{E}) \cap \overline{B} = \phi \Rightarrow \tilde{E} \bar{\delta} B$.

Therefore δ is a proximity on X. So it induces a topology τ_δ on X.

Let A be any subset of X. Denote by $\overline{A}$ and $C(A)$ respectively the τ-closure and τ_δ-closure of A.

$x \in C(A) \Rightarrow x\ \delta\ A \Rightarrow \overline{\{x\}} \cap \overline{A} \neq \phi \Rightarrow \{x\} \cap \overline{A} \neq \phi$. $\left[\because \overline{\{x\}} = \{x\}\right]$

$\Rightarrow x \in \overline{A}$.

Again, $x \in \overline{A} \Rightarrow \{x\} \cap \overline{A} \neq \phi \Rightarrow \overline{\{x\}} \cap \overline{A} \neq \phi$

$\Rightarrow x\ \delta\ A \Rightarrow x \in C(A)$.

Hence $C(A) = \overline{A}$. $\Rightarrow \tau_\delta = \tau$.

Therefore δ is compatible with τ.

Lemma 2 : Let (X, δ) be a proximity space and let A, B be any two subsets of X.

(i) $A \bar{\delta} X \backslash B \Rightarrow A \bar{\delta} X \backslash \text{int}(B)$

(ii) $A \bar{\delta} X \backslash B \Rightarrow \overline{A} \subset \text{int}(B)$.

(iii) Let $A \bar{\delta} X \setminus B$. Then there is a subset C of X such that $A \bar{\delta} X \setminus C$ and $C \bar{\delta} X \setminus B$, where closure and interior are taken with respect to $\tau_{(\delta)}$.

Proof : (i) Let $A \bar{\delta} X \setminus B$. Then there is a subset E of X such that

$$A \bar{\delta} E \text{ and } \tilde{E} \bar{\delta} (X \setminus B) \quad \ldots \ldots (1)$$

where $\tilde{E} = X \setminus E$.

Let $x \in \mathrm{cl}(X \setminus B)$. Then $x \delta (X \setminus B)$. If $x \in \tilde{E}$, then $\tilde{E} \delta x$ and so $\tilde{E} \delta (X \setminus B)$ which contradicts (1).

Hence $x \notin \tilde{E} \Rightarrow x \in E$; so $\mathrm{cl}(X \setminus B) \subset E$.

This with (1) implies that $A \bar{\delta} \overline{(X \setminus B)}$.

Now $\overline{X \setminus B} = X \setminus \mathrm{int(B)}$

So $A \bar{\delta} [X \setminus \mathrm{int}(B)] \quad \ldots \ldots (2)$

(ii) Let $A \bar{\delta} (X \setminus B)$. Then (2) holds. Take any x in $\bar{A}$. This with (2) implies that

$x \notin [X \setminus \mathrm{int}(B)]$; so $x \in \mathrm{int}(B)$.

Hence $\bar{A} \subset \mathrm{int}(B)$.

(iii) Let $A \bar{\delta} (X \setminus B)$. Then there is a subset E of X such that (1) holds. Write $C = X \setminus E$.

Then $C = \tilde{E}$ and $E = X \setminus C$. From (1) we have $A \bar{\delta} (X \setminus C)$ and $C \bar{\delta} (X \setminus B)$.

Theorem 4 : Let (X, δ) be a proximity space. Then the topology τ_δ is completely regular.

Proof : Let A be a closed set in X with respect to the topology τ_δ and $x \in X \setminus A$. Write $U_0 = X \setminus A$. Then U_0 is an open set and $x \in U_0$.

Since $x \notin A = \bar{A}$, $x \bar{\delta} (X \setminus U_0)$. $[\because A = X \setminus U_0]$

So there is a set $E \subset X$ such that

$$x \bar{\delta} (X \setminus E) \text{ and } E \bar{\delta} (X \setminus U_0) \quad \ldots \quad \ldots \quad (1).$$

Write $U_{1/2} = \text{int}(E)$. Then $U_{1/2}$ is an open set. By (1) and Lemma 2,

$$x\bar{\delta}\ (X\backslash U_{1/2}) \text{ and } U_{1/2}\ \bar{\delta}\ (X\backslash U_0)$$

There are subsets E_1 and E_2 of X such that $x\bar{\delta}X\backslash E_1$, $E_1\bar{\delta}\ X\backslash U_{1/2}$ and $U_{1/2}\ \bar{\delta}\ X\backslash E_2$, $E_2\bar{\delta}$ $X\backslash U_0$ ... (2)

Write $U_{1/4} = \text{int}(E_1)$ and $U_{3/4} = \text{int}(E_2)$.

Then $U_{1/4}$ and $U_{3/4}$ are open sets. By (2) and Lemma 2 we get.

$x\bar{\delta}\ (X\backslash U_{1/4})$, $U_{1/4}\ \bar{\delta}(X\backslash U_{1/2})$ and

$$U_{1/2}\ \bar{\delta}(X\backslash U_{3/4}),\ U_{3/4}\ \bar{\delta}\ (X\backslash U_0) \ \ldots \quad \ldots \ (3)$$

So, $x\in U_{1/4}\subset\bar{U}_{1/4}\subset U_{1/2}\subset\bar{U}_{1/2}\subset U_{3/4}\subset\bar{U}_{3/4}\subset U_0$.

Denote by D the set of numbers of the form

$$\frac{m}{2^n} : m = 1,3,5,\ldots,2^{n-1} \text{ and } n = 1,\ 2,\ 3,\ \ldots..$$

Then D is dense in $[0,1]$.

Proceeding as above we can select a family of open sets $\{U_t : t \in D\}$ such that if t, $s \in D$ and $t < s$, then

$$x\in U_t\subset\bar{U}_t\subset U_s\subset\bar{U}_s\subset U_0.$$

We now define the functions f on X as follows :

Let $z\in X$. $f(z) = 0$ if $z \in\cap\{U_t : t\in D\}$

$= 1$ if $z \in X\backslash U_0$

$= \inf\ \{t : z \in U_t\}$ otherwise.

Clearly values of f lie in the closed interval $[0,\ 1]$ and $f(x) = 0$ and $f(z) = 1$ for all $z \in A$.

Now we prove that f is continuous.

The family of all intervals $[0,\ a)$, $(b,\ 1]$ $(0 < a,\ b < 1)$ forms a subbase for the topology on $[0,\ 1]$. It is easy to see that

$$f(z) < a \Leftrightarrow z \in U_t \text{ for some } t \text{ in } D \text{ with } t < a.$$

This gives that $f^{-1}\ ([0,\ a)) = \cup\{U_t : t \in D \text{ and } t < a\}$.

So $f^{-1}\ ([0,\ a))$ is open. Again,

$f(z) > a \Leftrightarrow z$ lies outside of $\bar{U}_t$ for some t in D with $t > a$.

So $f^{-1}((a, 1]) = \cup \{X \setminus \bar{U}_t : t \in D \text{ and } t > a\}$.

Which is open.

This gives that f is continuous.

Hence the space (X, τ_δ) is completely regular. This proves the theorem.

Theorem 5 : Let (X, τ) be a completely regular space and let it have a compatible proximity δ defined as follows : For any two subsets A, B of X.

$$A \,\delta\, B \text{ iff } \bar{A} \cap \bar{B} \neq \phi. \text{ Then } (X, \tau) \text{ is normal.}$$

Proof : Let A and B be any two disjoint closed subsets of X. Then $A \bar{\delta} B$. So there is a subset E of X such that

$$A \bar{\delta} E \text{ and } \tilde{E} \bar{\delta} B \quad \ldots \ldots (1)$$

Where $\tilde{E} = X \setminus E$.

By Lemma 2, we have

$$A \subset \text{int}(X \setminus E) \text{ and } B \subset \text{int}(X \setminus \tilde{E}) = \text{int } E. \ldots\ldots (2)$$

Write $G_1 = \text{int}(X \setminus E)$ and $G_2 = \text{int}(E)$.

Then G_1 and G_2 are open. Clearly $G_1 \cap G_2 = \phi$.

By (2) $A \subset G_1$ and $B \subset G_2$.

Hence the space (X, τ) is normal.

Exercise : Prove that the interior of a set. $A \subset X$ endowed with a uniformity $\mathscr{V}$ is

$\mathring{A} = B = \{x \in X : V(x) \subset A \text{ for some } V \in \mathscr{V}\}$

Solution : Since every open set $G \subset A$ is contained in B, so it is sufficient to prove that the set B is open. Take any $x \in B$. Then there is a $V \in \mathscr{V}$ such that $V(x) \subset A$. Choose $W \in \mathscr{V}$ such that $W \circ W \subset V$. Note that for any $y \in W(x)$, if $z \in W(y)$ then $(x, y) \in W$ and $(y, z) \in W$ and so $(x, z) \in W \circ W \subset V \Rightarrow Z \in V(x)$. Thus

$$W(y) \subset V(x) \subset A.$$

Since this is true for every $y \in W(x)$ this shows that $W(x) \subset B$. So x is an interior point of B. As x is arbitrary, this proves that B is open.

Exercise : Let X be a Tychonoff space. Let $C(X)$ and $C^*(X)$ denote the family of all real valued continuous functions and real valued continuous bounded functions on X respectively. For every finite number of functions $f_1, f_2, ..., f_k \in C(X)$ (or $C^*(X)$)

$$d_{f_1, f_2, ..., f_k}(x,y) = \max_{1 \le i \le k} \{| f_i(x) - f_i(y)|\}$$

define two pseudometrics on X.

Solution : It is easy to note that

$$d_{f_1, f_2 ..., f_k}(x,x) = \max\{|f_1(x) - f_1(x)|, ..., | f_k(x) - f_k(x)|\}$$

$$= 0 \quad \forall x \in X.$$

If $x, y \in X$ then since $|f_i(x) - f_i(y)| = |f_i(y) - f_i(x)|$

for $i = 1, 2, .., k$ so it follows that

$$d_{f_1, f_2 ... f_k}(x,y) = d_{f_1, f_2 ... f_k}(y,x)$$

Finally if $x, y, z \in X$ then we have

$$|f_i(x) - f_i(z)| \le | f_i(x) - f_i(y)| + | f_i(y) - f_i(z)|$$

for $i = 1, 2, ...k$. Hence

$$d_{f_1, f_2 ... f_k}(x,z) \le d_{f_1, f_2, ..., f_k}(x,y) + d_{f_1 ... f_k}(y,z).$$

Exercise : Let P and P^* denote the families of pseudometrics on X defined as in the preceeding exercise. Consequently they generate uniformities $\mathscr{V}$ and $\mathscr{V}^*$ on X. Prove that $\mathscr{V}$ and $\mathscr{V}^*$ induce the same topology identical with the initial topology.

Solution : Since any $f_1, f_2, .., f_k \in C(X)$ (or $C^*(X_i)$) are continuous and the modulus function is continuous so every generated pseudometric

$d_{f_1, f_2 ... f_k}(x,y) = \max \{|f_1(x) - f_1(y)|, .., |f_k(x) - f_k(y)|\}$ is a continuous function from $X \times X \to R$. Thus every element of P (or P^*) is a continuous function from $X \times X \to R$. Hence the sets $\{(x, y) : d(x, y) < z^{-i}\}$ where $d \in P$ and $i \in N$ are open in $X \times X$ and every open set in the topology induced by $\mathscr{V}$ or $\mathscr{V}^*$ is open in the initial topology on X.

Now let us suppose that U is open in the initial topology in X. Let $x_0 \in U$. Since

X is a Tychonoff space, there is a continuous function f, i.e. $f \in C^*(X) \subset C(X)$. Such that $f(x_0) = 0$ and $f(x) = 1$ for $x \in X \backslash U$. Let

$$V = \left\{(x,y) : d_f(x,y) < \frac{1}{2}\right\}.$$

Then we have $V(x_0) \subset U$ and this implies that U is open in the topology induced by $\mathscr{V}$ or $\mathscr{V}^*$.

Exercise : Let X be a Tychonoff space, $C^*(X)$ denote the family of real valued bounded continuous functions on X, p^* the family of generated pseudometrics. Then $(X, \mathscr{V}^*)$ is totally bounded where $\mathscr{V}^*$ is the uniformity generated by p^*.

Solution : It is sufficient to prove that for every system of functions $f_1, f_2, \ldots, f_k, \ldots \in C^*(X)$ and $\in > 0$, there exists a finite number of points $x_1, x_2, \ldots, x_n \in X$ such that for every $x \in X$, there exists an $i \leq n$ with the property.

$$d_{f_1 f_2 \ldots f_k}(x, x_k') = \max\{| f_1(x) - f_1(x_i)|, \ldots, | f_k(x) - f_k(x_i')|\} < \in.$$

Since $f_1, f_2 \ldots, f_k \in C^*(X)$ so $f_1(X), f_2(X), \ldots, f_k(X)$ are all bounded sets in R and so we can find a bounded closed interval $J \subset R$ which contains $f_1(X), \ldots, f_k(X)$. Note that J is totally bounded and so we can find a finite number of open intervals $\{A_j\}_{j=1}^{m}$ of diameter less than $\in$ which cover J. Subsequently the family of sets of the form

$$f_1^{-1}(A_{j_1}) \cap f_2^{-1}(A_{j_2}) \cap \ldots \cap f_k^{-1}(A_{j_k}), \quad \ldots\ldots\ldots(1)$$

where $1 \leq j_i \leq m$ for every $i \leq k$ is a covering of the space X. The diameter of each of these sets with respect to the pseudometric $d_{f_1 f_2 \ldots f_k}$ is less than $\in$. Choosing a point x_i from each of the non-empty sets of the form (1) we get the finite sequence of points $x_1, x_2, \ldots, x_n$ which has the required property.

Group–A

(Short questions)

1. Describe the uniformity on the real number space which induces the usual topology on R and the uniformity which induces the discrete topology on R.
2. If the uniformity $\mathscr{V}$ on a set X has a countable base then show that the induced topology is first countable.

3. If the intersection of all members of the uniformity consists of the diagonal only then prove that the induced topology is T_1.
4. In an uniform space give an example of a set which is totally bounded but not compact.
5. Give an example of a continuous mapping from a uniform space $(X, \mathscr{V})$ to another uniform space $(Y, \mathscr{V})$ that is not uniformly continuous.
6. Give an example of a topological space which is not uniformizable.
7. If a uniform space $(X, \mathscr{V})$ is complete then prove that the space $(M, \mathscr{V}_M)$ is complete for each closed set $M \subset X$.
8. If v_1 and v_2 are two uniformities on X and $v_1 \supset v_2$ then show that v_1 induces a stronger topology than the topology induced by v_2.

Group—B
(Long Questions)

1. Let $(X, \mathscr{V})$ and $(Y, \mathscr{V})$ be uniform spaces. Prove that $f : (X, \mathscr{V}) \to (Y, \mathscr{V})$ is uniformly continuous iff for every $V \in B'$ there is a $U \in B$ such that $U \subset f^{-1}(V)$ where B and B' are bases of $\mathscr{V}$ and V respectively.
2. Show that every family $\{V_s\}_{s\in\Delta}$ of uniformities in a set X has a least upper bound i.e. in the set X there exists a uniformity V which is weakest in the set of all uniformities stronger than V_s for every $s \in \Delta$.
3. Let (X, V) be a uniform space, verify that the product of the topology induced by V on $X \times X$ is identical with the topology induced by $V \times V$ in $X \times X$.
4. If a uniformity V in a set X is induced by a metric ρ then prove that (X, V) is complete if the metric space (X, ρ) is complete.
5. Let (X, V) be a uniform space and let (Y, V) be a complete uniform space. Show that every uniformly continuous funciton f defined on (A, V_A) where A is a dense subset of x, with values in (Y, V) can be extended to a uniformly continuous function from (X, V) to (Y, V).
6. Let X be a compact space. Show that there exists exactly one uniformity V in X which induces the topology on X. The base for the uniformity V consists of all neighbourhoods of the diagonal which are open in the space $X \times X$.
7. Prove that the filter associated with a Cauchy net is a Cauchy filter and conversely every net associated with a Cauchy filter is cauchy.

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Notes

মানুষের জ্ঞান ও ভাবকে বইয়ের মধ্যে সঞ্চিত করিবার যে একটা প্রচুর সুবিধা আছে, সে কথা কেহই অস্বীকার করিতে পারে না। কিন্তু সেই সুবিধার দ্বারা মনের স্বাভাবিক শক্তিকে একেবারে আচ্ছন্ন করিয়া ফেলিলে বুদ্ধিকে বাবু করিয়া তোলা হয়।

—রবীন্দ্রনাথ ঠাকুর

ভারতের একটা mission আছে, একটা গৌরবময় ভবিষ্যৎ আছে, সেই ভবিষ্যৎ ভারতের উত্তরাধিকারী আমরাই। নূতন ভারতের মুক্তির ইতিহাস আমরাই রচনা করছি এবং করব। এই বিশ্বাস আছে বলেই আমরা সব দুঃখ কষ্ট সহ্য করতে পারি, অন্ধকারময় বর্তমানকে অগ্রাহ্য করতে পারি, বাস্তবের নিষ্ঠুর সত্যগুলি আদর্শের কঠিন আঘাতে ধূলিসাৎ করতে পারি।

—সুভাষচন্দ্র বসু

Any system of education which ignores Indian conditions, requirements, history and sociology is too unscientific to commend itself to any rational support.

—Subhas Chandra Bose

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NETAJI SUBHAS OPEN UNIVERSITY

STUDY MATERIAL

MATHEMATICS

POST GRADUATE

PG (MT) - IX B(II)
(APPLIED MATHEMATICS)

Mathematical Models
in Ecology

PREFACE

In the auricular structure introduced by this University for students of Post- Graduate degree programme, the opportunity to pursue Post-Graduate course in Subject introduced by this University is equally available to all learners. Instead of being guided by any presumption about ability level, it would perhaps stand to reason if receptivity of a learner is judged in the course of the learning process. That would be entirely in keeping with the objectives of open education which does not believe in artificial differentiation.

Keeping this in view, study materials of the Post-Graduate level in different subjects are being prepared on the basis of a well laid-out syllabus. The course structure combines the best elements in the approved syllabi of Central and State Universities in respective subjects. It has been so designed as to be upgradable with the addition of new information as well as results of fresh thinking and analysis.

The accepted methodology of distance education has been followed in the preparation of these study materials. Co-operation in every form of experienced scholars is indispensable for a work of this kind. We, therefore, owe an enormous debt of gratitude to everyone whose tireless efforts went into the writing, editing and devising of a proper lay-out of the materials. Practically speaking, their role amounts to an involvement in invisible teaching. For, whoever makes use of these study materials would virtually derive the benefit of learning under their collective care without each being seen by the other.

The more a learner would seriously pursue these study materials the easier it will be for him or her to reach out to larger horizons of a subject. Care has also been taken to make the language lucid and presentation attractive so mat they may be rated as quality self-learning materials. If anything remains still obscure or difficult to follow, arrangements are there to come to terms with them through the counselling sessions regularly available at the network of study centres set up by the University.

Needless to add, a great deal of these efforts is still experimental-in fact, pioneering in certain areas. Naturally, there is every possibility of some lapse or deficiency here and there. However, these do admit of rectification and further improvement in due course. On the whole, therefore, these study materials are expected to evoke wider appreciation the more they receive serious attention of all concerned.

Professor (Dr.) Subha Sankar Sarkar
Vice-Chancellor

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NETAJI SUBHAS OPEN UNIVERSITY

PG(MT) : IX B(II)

Mathematical Models in Ecology

Unit 1 ❑ Introduction

Objectives : The object of this chapter is to present the basic concepts of ecology along with the mathematical modeling of ecological system.

Structure

1.1 ❑ Ecology : Basic Concepts

Definition : Environment:

The place where a living organism lives with its surrounding form its environment. Environment consists of two parts : abiotic and biotic. Soil, water, air and different minerals form the abiotic (or physical) environment, where as the biotic environment is formed of the plants and animals. The living organisms and environment are interrelated.

Definition : Ecology

The branch of science which deals with the study of interrelationship among the living organisms in relation with the environment is known as ecology. German biologist E. Haeckel (1968) first introduced the term 'ecology', which is derived from the Greek word 'Oikos' meaning dwelling place or house and 'logy' meaning the study of.

Parts of Ecology:

The study of ecology consists of four parts : (i) individual (ii) population (iii) community (iv) ecosystem. We describe them separately.

(i) Individual : It deals with the study of growth, development, reproduction and mortality of an individual.

(ii) Population : It deals with the study of the problems of the different organisms of the same single species. It studies whether a population will grow or decline, it studies why some populations are stable over many generations while other show outbreaks and crashes, it studies the causes of extinction.

(iii) Community : It deals with populations of different species. The problems to be studied are whether populations of different species co-exist ? Do the details of feeding relationship (who eats whom) matter ?

(iv) Ecosystem : Ecosystem is the fundament unit of ecology, where both biotic and abiotic components of the environment interplay. An ecosystem consists of several factors which may be divided into two categories : abiotic and biotic.

Components of Ecosystem:

Abiotic factors :

(i) **Different organic and inorganic components** : Calcium, sulpher, magnesium, potassium, oxygen, nitrogen, carbon dioxide, water, soil, amino-acids etc.

(ii) **Physical factors** : Light, humidity, temperature, atmospheric pressure, rainfall etc.

(iii) **Soil factor** : Nature of soil, water holding capacity, percolation of water through soil etc.

(iv) **Topographic factors** : Altitude, undulating landscape, amount of light falling on a place, wind blowing through etc.

Biotic factors:

(i) **Producers** : Green plants which produce proteins, amino acids, glucose etc. by the process of photosynthesis in the presence of sun-light,

(ii) **Consumers:**

(a) Primary consumers : Plant eaters - minute animals in the upper level of water constitute zoo plankton e.g. paphia, protozoa. The primary consumers in the lower level of water are called bottom forms e.g. orthopods, snails, small fishes, etc. Primary consumers of land are harvivores e.g. grasshopper, rabbits, monkey, dears, cows etc.

(b) Secondary consumers : Carnivores feeding on primary consumers such as frogs, toads, spider etc.

(c) **Territory consumers** : The animals feeding on the secondary consumers are called territory consumers e.g. Tiger, Lion, Leopard, whale, hawk, eagle etc.

(iii) Decomposers : (also called microorganisms) Certain bacteria, fungi breakdown the complex compounds of dead protoplasm, absorb certain decomposed produced and release certain simple substance for further utilization by the producers.

Different Types of Ecosystems :

(i) **Aquatic Ecosystems** : A pond is an example of an aquatic ecosystem. It comprises of four components : abiotic factors, producers, consumers and decomposers,

(ii) **Terrestrial Ecosystem** : A forest is a typical example. It also comprises of four components : abiotic substances, producers, consumers and decomposers.

1.2 ❑ Ecological Systems : Mathematical Models

Much of the monograph is devoted to the formulation and analysis of mathematical models. A mathematical model is a set of assumptions about an ecological system expressed in mathematical language. Mathematical reasonings or computations may then be used to generate predictions about the system.

Definition : Dynamical Model

A dynamical model of a system is a mathematical statement of the rules governing the changes of the states or conditions of the system with time. A dynamical model may be deterministic or stochastic. A dynamical model may be discrete-time or a continuous time.

1.2.1 Deterministic Models and State variables

The simplest ecological models, called deterministic models, make assumption that if we know the present conditions of a system, we can predict its future accurately. To determine the current state or condition of the system we have to choose some quantities called state variables. The choice of state variables involve a subtle balance of biological realism and mathematical complexity.

(i) State variables for individual are age, sex, development stage, physiological variable such as weight or size. For many cases age and size (or weight) are sufficient to serve as state variables.

(ii) State variables for populations are the number of living organisms the population contains. A more general population is a structured population. Structured populations are of two types : (a) age-structured population which involves both the number of individuals and their ages as state variables (b) spatially-structured population which involves the number of individuals along with their positions or locations at any time.

(iii) State variables for a community (a group of populations of different species) are the number of individuals of each dynamically interacting species.

(iv) In ecosystem the species are divided into some functional groups such as primary producers, herbivores, carnivores having interaction among the groups. Workable state variables to an ecosystem is a list of the biomasses of each of the functional group.

1.2.2a Modelling in Discrete-time

Let the variable X_t denotes the state of the system at time t. The system state at time $t + \Delta t$ denoted by $X_{t+\Delta t}$ is a function of X_t i.e.

$$X_{t+\Delta t} = F(X_t) \qquad \ldots \qquad (2.1)$$

The functional form of F depends on the system under consideration. If the function F is explicitly independent of time t the equation (2.1) is called an autonomous difference equation. The difference equation model forecasts the state of the system at series of equally spaced times. For example, if we know the state at time t = 0, we can calculate its state at times $t = \Delta t, 2\,\Delta t, 3\,\Delta t \ldots$; Δt represents a single number, say one second, one minute, one year etc. For non-autonomous systems the difference equation is of the form

$$X_{t+\Delta t} = F(X_t, t) \qquad \ldots \qquad (2.2)$$

1.2.2.b Modelling in Continuous—time

Continuous—time models aim to predict the values of the state variables at all future time, not at integer multiples of some time increment Δt. To write down the dynamics of a system we require the rate of change of the state variable X. It can be written in the form of the non-autonomous differential equation

$$\frac{dX}{dt} = g(X,t) \qquad \ldots \qquad (2.3)$$

For autonomous system the rate of change of the state variable that is, the function g does not depend explicitly on t. In this case the equation becomes

$$\frac{dX}{dt} = g(X) \quad \cdots \quad (2.4)$$

1.2.3 Balance (or conservation) Equation

Changes in abundance, stock or concentration of any physical or biological entity occur only through the operation of an identifiable process. For example, the concentration of physically and chemically stable material in an enclosed region can only change because of the import or export across the boundaries of the region. If the system is reactive then we must add the possibility of chemical transformation. Similarly, the population of organisms in an enclosed region can only change because of reproduction, mortality, export and import of the population across the boundaries. The dynamical equation which represents the changes in mathematical language is called the conservation or balance equation.

(a) Balance Equation for Chemically Inert Substances

We consider a chemically non-reactive substance located within a region of space. Let Q_t represents the quantity of the substance within the region at any time t. Then the balance equation is given by

$$Q_{t+\Delta t} = Q_t + \text{inflow} - \text{outflow} \quad \cdots \quad (2.5)$$

where the terms 'inflow' and 'outflow' represent the total inflow and outflow of the material during the time interval $(t, t+\Delta t)$. The equation (2.5) is an example of discrete-time balance equation. The analogue equation for continuous time is

$$\frac{dQ(t)}{dt} = \text{inflow rate} - \text{outflow rate} \quad \cdots \quad (2.6)$$

(b) Balance Equation for Chemically reacting substances

In the above balance equations the stock changes because of transport into and out of the region of interest. Most ecologically interesting situations involve chemical and biological transformation within the region being modeled. For example, the balance equation for a chemically reacting system is

$$\frac{dQ(t)}{dt} = \text{inflow rate} - \text{outflow rate} + \text{formation rate} - \text{transformation rate} \quad \cdots \quad (2.7)$$

In conclusion, the deterministic models of ecological systems involves three steps :

(i) Choose the state variables appropriate to the system under consideration,

(ii) Derive the balance (or conservation) equations. The balance equations represent the model equations for the growth process of the system under consideration,

(iii) For the successful utilization of the model equations we have to make model-specific assumptions.

1.2.4 Randomness and Stochastic Models

In the deterministic models the state of a system at any future time can be predicted exactly from its present state. This assumption is of course untenable. Unpredictability or randomness enter ecological dynamics in two ways. First no environment outside the laboratory is truely predictable. For example, the average light intensity measured each day at place vary randomly. Since light provides the energy for primary production the dynamics of ecological system will be seriously affected by the variability. Similarly, the random variation of humidity, temperature and other factors for an ecosystem can not be correctly predicted by deterministic models.

A second important way in which randomness affects ecological dynamics is that similar organisms do not necessarily respond in the same way to a given environment. Genetically identical individuals with identical histories in identical environment exhibit considerable variability in the timing, amount of reproduction and mortality. Although randomness is ubiquitous and stochastic models are essential, deterministic models are appropriate starting point for many ecological systems and are prerequisite to the formulation, analysis and better understanding of stochastic models of complex systems under investigation. In this monograph we shall be concerned mainly with the deterministic dynamical models of ecological systems.

1.2.5. Summary

The chapter consists of two parts:

(i) The first part consists of a brief discussion of ecology and ecological systems,

(ii) The second part is concerned with the dynamical modeling of ecological system.

It explains the concepts of state variables, continuous-time and discrete-time models of ecological systems The difference between deterministic and stochastic models of ecological system has been explained.

Unit 2 ❑ Single-Species Population : Continuous-Time Models

Objectives: The object of this chapter is to present the basic biological and mathematical postulates necessary for the continuous-time models of single-species populations together with their mathematical analysis.

Structure

2.1 ❑ Introduction : Basic Postulates

For the development of continuous-time models of population we assume the following three biological and mathematical postulates :

(i) The postulate of Parenthood

This states that every living organism has arisen from at least one parent of like kind to itself; it is often called 'the principle of a biogenesis'. For any one who believes in the initial terrestrial origin of life, the postulate is not universally valid; but since under present condition spontaneous generation has never been observed, we can take it as true enough to use in our investigation.

(ii) The postulate of upper limit

The second postulate is that in a finite space there is an upper limit to the number of living beings that can in some way occupy or utilize the space under consideration. The living beings require supply of energy at a certain rate to maintain their stability; obviously the space can not contain more of such living beings that utilize the energy input in the space.

(iii) The postulate of continuity

In addition to the above two biological postulates, it is convenient for mathematical reasons initially to adopt the convention that the variation of a population size x behaves as if x is a continuous variable, capable of taking any value, integral, fractional between the possible upper and lower limits of the population. The convention, though strictly untrue, is harmless when we are dealing with a sufficiently large population not having definite breeding or dying seasons, in which reproduction occurs at random among all members of the appropriate age class, and death occurs according to some statistically defined pattern not varying with time. When definite breeding seasons occur, or when mortality is much greater at sometimes of the year than at others, finite difference equations are to be used.

2.2 ❑ Population Growth : General Model Equation

Let x(t) denotes the population size (or density) at any time t and according to the postulate (iii) x(t) is assumed to be differentiable every where, that is, a smooth function of time t. The general model equation of growth of a single-species population can be written as

$$\frac{dx}{dt} = f(x) \qquad \ldots \qquad (2.1)$$

where the growth rate $\frac{dx}{dt}$ depends only on the population size (or density). Such an assumption appears to be reasonable for simple organisms such as microorganisms. For more complicated organisms like animals or humans this is an over simplification as it ignores intra-species competitions for resources and other factors, including age structure (the morality rate may depend on age rather than on population size). If the function f is sufficiently smooth, we can expand it in Taylor's series,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \qquad \ldots \qquad (2.2)$$

The postulate (i) requires f(o) = 0 to dismiss the possibility of spontaneous generation, the production of living organisms from inanimate matter. This is equivalent to

$$\left.\frac{dx}{dt}\right|_{x=0} = f(o) = 0$$

so that we may assume $a_0 = 0$ and then

$$\frac{dx}{dt} = a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$= x(a_1 + a_2 x + a_3 x^2 + ...)$$
$$= x\, g(x) \quad \dots \quad (2.3)$$

The quantity $\frac{1}{x}\frac{dx}{dt} = \frac{x'(t)}{x(t)}$ is called the per capita growth rate i.e. the rate of growth per member. It is also known as intrinsic growth rate and the polynomial g(x) of (2.3) is, therefore, the intrinsic growth rate of the population. We shall now study several specific growth models and study their characteristic behaviours.

2.3 ❑ Malthus Population Model: Exponential Growth

We first look at a population in which all individuals develop independently of one another. The organisms live in an unrestricted environment, where there is no form of competition. The rate of change of populations size (or density) can be computed if the birth-rate, death-rate and migration rate are known. For a closed population system there is no migration and the population size changes due to changes in births and deaths. Let b be the per capita birth rate and d be the per capita death rate. Then the change in population during a small time-internal (t, t + h) is given by

$$x(t + h) - x(t) \simeq (b - d)\, x(t)\, h$$

$$\text{or, } \frac{x(t+h) - x(t)}{h} \simeq (b - d)\, x(t)\, h$$

Taking the limit $h \to 0$, we have,

$$\frac{dx}{dt} = (b - d)\, x = rx \quad \dots \quad (2.4)$$

$$\text{where } \quad r = (b - d) \quad \dots \quad (2.5)$$

is the net growth (or reproduction) rate.

The equation (2.4) is the famous Malthus model equation of population growth. This is the simplest form of the general model equation (2.3) with coefficients of g(x) as

$$a_1 = r, a_2 = a_3 = ... = 0 \quad \dots \quad (2.6)$$

The equation (2.4) can be solved to give the exponential distribution

$$N(t) = N_0 e^{rt} \quad \dots \quad (2.7)$$

where $N_0 = N(o)$, the initial population. For this reason, the population obeying the equation (2.4) is said to be undergoing exponential growth. This constitutes the simplest minimal model of bacterial growth or indeed growth of any reproductive population. It was first initiated by Malthus in the year 1798 in human populations in a treatise that caused sensation in the scientific community of the day. He (Malthus) claimed that, barring natural disasters, the world's population would grow exponentially and thereby eventually outgrow its resources. He concluded that mass starvation would befall huminity.

The equation (2.4), while very simple, turns up in a number of natural processes. By reversing the sign of r one obtains a model of a population in which a fraction r of the individuals is continually removed per unit time, such as by death or migration. The equation

$$\frac{dN}{dt} = -rx \qquad \ldots \qquad (2.8)$$

with solution $N(t) = N_0 e^{-rt}$ ... (2.9)

describes a decaying process. This equation is commonly used to describe radioactive decay.

One can define a population doubling time τ_2 (for $r > 0$) or half-life τ_1 (for $r < 0$) in the following way. For growing population, seek a time t_2 such that

$$\frac{N(\tau_2)}{N_o} = 2$$

Putting this in equation in (2.7), we obtain

$$\frac{N(\tau_2)}{N_o} = 2 = e^{e\tau}$$

or, $\ln 2 = r\tau_2$ or $\tau_2 = \dfrac{\ln 2}{r}$ ... (2.10)

The doubling time τ_2 is thus inversely proportional to the reproductive constant r. In a similar way we can find the half-life of a decaying population.

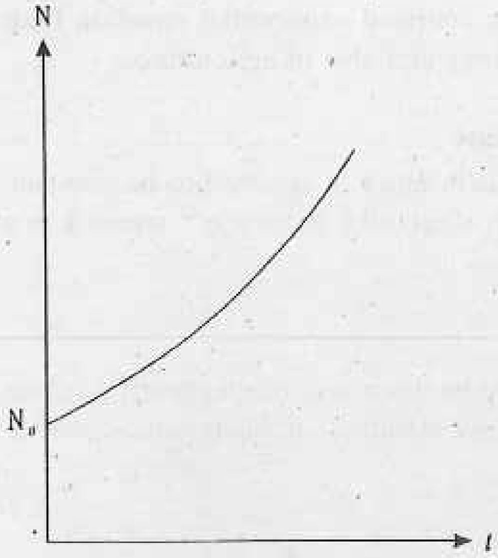


Fig. 2.3a

Malthus law for exponential growth

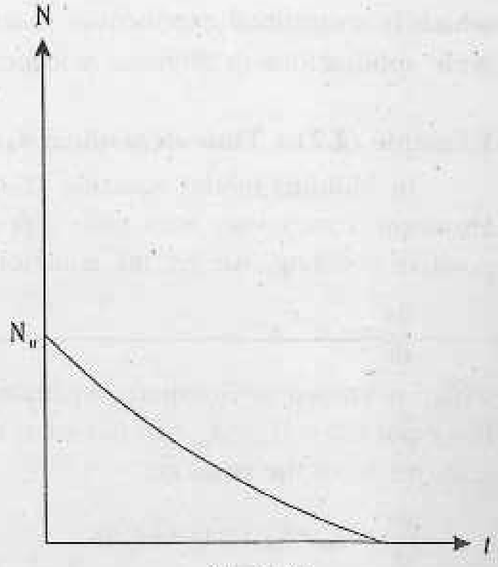


Fig. 2.3b

Malthus law for exponential delay

Remark:

The model (2.4) is not accurate for all time. Populations that grow exponentially at first are commonly observed in nature. However, their growth rates usually tend to decrease as population size increases. In fact, exponential growth or decay may be considered typical local behaviour. In other words, populations dynamics can usually be approximated by this simple model only for a short period of time. The assumption that the rate of growth of a population is proportional to its size (linear assumption) is unrealistic on long time scales. In the next section we shall modify this model for a realistic population growth. Note that a population that grows exponentially to infinity violates our basic postulates of finite upper limit.

Example (2.1): Confined Exponential Distribution

In order to prevent the infinitely large population size in Malthus exponential growth model we replace it by a confined exponential growth model equation as

$$\frac{dx}{dt} = r\,(x^* - x), \qquad x(o) = x_o$$

where x^* is the equilibrium value of x.

With initial condition $x(o) = x_o$, the solution of the equation is

$$x = x^* - (x^* - x_0)\, e^{-rt}$$

which is a confined exponential function. The confined exponential equation finds wide applications in physical science, technology and also in agriculture.

Example (2.2) : Time-dependent Growth Rate

In Malthus model equation (2.4) the growth rate r is assumed to be constant. However, r may vary with time. For example, if we rake $r(t) = r_0 e^{-kt}$ where k is a positive constant, we get the equation

$$\frac{dx}{dt} = r_0 e^{-kt} x$$

which is known as Gompertz equation and is to be discussed independently later on. If we put $r(t) = [r_0 + r_m \operatorname{Sin}(wt + \phi)]$ that is, if we assume a simple harmonic growth rate, we have the equation

$$\frac{dx}{dt} = [r_0 + r_m \operatorname{Sin}(wt + \phi)]x.$$

This type of equation is useful for the certain types of trees whose mass vary periodically with a period of one year on the average.

2.4 ❑ Logistic Population Model

To correct prediction based on Malthus model or law (that a population grows indefinitely at an exponential rate), we consider a non-constant intrinsic growth rate g(x). The logistic model is perhaps the simplest extension of Malthus model equation (2.4). For a faithful model of population growth, we take more terms in the series for f(x). We take the intrinsic growth rate as

$$g(x) = a_1 + a_2 x = r\left(1 - \frac{x}{k}\right) \qquad \dots \qquad (2.11)$$

$$\text{where } a_1 = r, \quad k = -\left(\frac{a_1}{a_2}\right) \qquad \dots \qquad (2.12)$$

The growth equation (2.3) then becomes,

$$\frac{dx}{dt} = r\,x\left(1 - \frac{x}{k}\right) \qquad \dots \qquad (2.13)$$

which is the famous logistic model equation of population growth.

Carrying Capacity :

From equation (2.13) we see that

$$\frac{dx}{dt} = 0 \qquad \text{when } x = k.$$

Thus x = k is a steady-state or equilibrium state of the logistic equation (2.13). We also note that

$$\left.\begin{aligned} \frac{dx}{dt} > 0, &\qquad \text{for } x < k \\ \frac{dx}{dt} < 0, &\qquad \text{for } x > k \end{aligned}\right\} \qquad \ldots \qquad (2.14)$$

The quantity k represents the carrying capacity of the environment of the species.

Example (2.3) : Solve the equation (2.13) i.e. the equation

$$\frac{dx}{dt} = r\,x\left(1 - \frac{x}{k}\right),\; x(o) = x_0 \qquad \ldots \qquad (2.13)$$

To solve we write the equation (2.13) in the form

$$\frac{dx}{x\left(1 - \frac{x}{k}\right)} = r\,d\,t$$

Rearrange the equation to show that the solution is given by

$$x(t) = \frac{x_0 k}{x_0 + (k - x_0)e^{-rt}} \qquad \ldots \qquad (2\text{-}15)$$

The solution (2.15) shows that for $t \to \infty$, the population size approaches the carrying capacity k. It is easy to show that when the initial population x_0 is very small, the population initially appears to grow exponentially at a rate r.

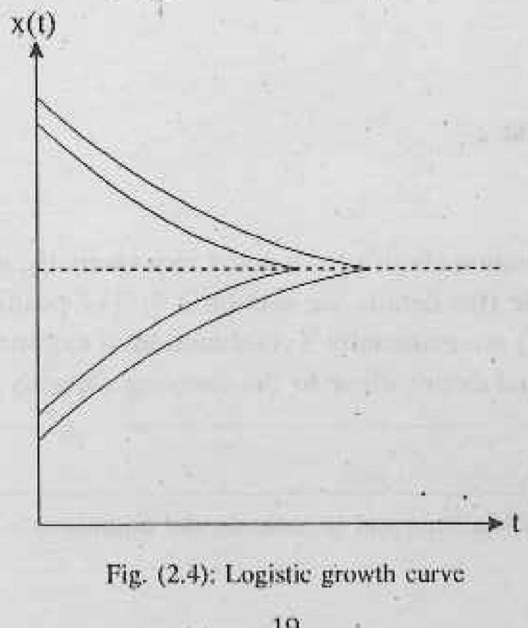


Fig. (2.4): Logistic growth curve

Intra-species Competition

The competition among the individuals of a species for limited food, habitat and other resources compel an increase in the net population mortality under crowded conditions. Such effects are prominent when there are frequent encounters between individuals. The equation (2.13) can be written as

$$\frac{dx}{dt} = r\,x - \frac{r}{k}x^2 \qquad \ldots \qquad (2.16)$$

The second term depicts a mortality proportional to the rate of paired encounters. The equation (2.13) is thus a modification of Malthus growth equation (2.4) by taking a term (the second term in the r.h.s of (2.16)) representing intra-species interaction which stops the exponential growth.

Behaviour near Equilibria

The logistic equation (2.13) has two equilibria, $x^* = 0$ and $x^* = k$.
Near $x^* = 0$, x^2/k is small compared to x so that

$$\frac{dx}{dt} \simeq r\,x \qquad \ldots \qquad (2.17)$$

For $r > 0$, small perturbation about $x^* = 0$ grows exponentially; the equilibrium $x^* = 0$ is unstable. Close to $x^* = k$, we put $y = x - k$ in equation (2.13) to give us

$$\frac{dx}{dt} = -r\,y - \frac{r}{K}y^2$$

Since y is small, we have

$$\frac{dx}{dt} \simeq -ry$$

For $r > 0$, small perturbation about $x^* = k$ decay exponentially, the equilibrium $x^* = k$ is asymptotically stable (for details see section 2.7). For positive r, solutions of the logistic equation (2.13) are essentially a combination of exponential growth, close to zero, and of exponential decay, close to the carrying capacity (see Fig. 2.4).

2.5 ❑ Allee Effect

A further extension of Malthus and logistic model equations is an assumption of the form

$$
\left.\begin{aligned} g(x) &= a_1 + a_2\, x + a_3\, x^2 \\ \text{with } a_2 &> 0 \text{ and } a_3 < 0 \end{aligned}\right\} \quad \ldots \quad (2.17)
$$

When this condition is satisfied we obtain Allee effect, which represents a population that has a maximal intrinsic growth rate at intermediate density. This effect may stem from the difficulty of finding mate at very low density. The Fig. (2.5) below is an example of density dependent form of g(x) that predicts the Allee effect. Its general characteristic can be summarized by the inequalities :

$$
\left.\begin{aligned} g'(x) &> 0, \qquad \text{for } x < \eta \\ g'(x) &< 0, \qquad \text{for } x > \eta \end{aligned}\right\} \quad \ldots \quad (2\text{-}18)
$$

where η is the density for optimal reproduction.

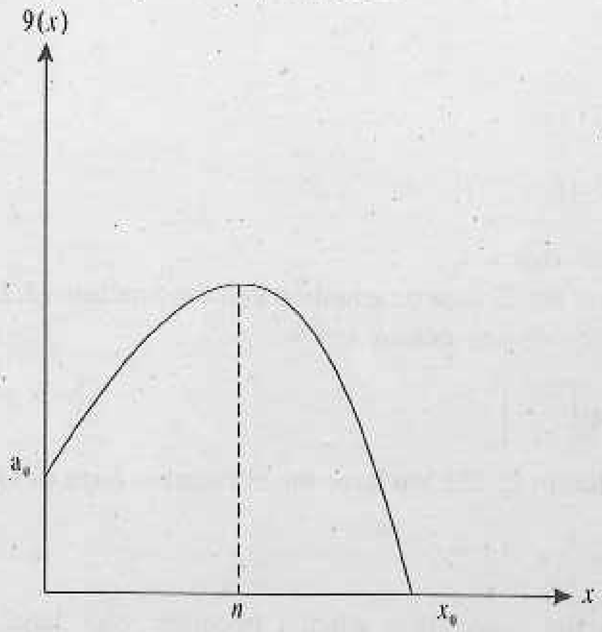


Fig. (2.5) : In the Allee effect the rate of reproduction is maximal at intermediate density

2.6 ❑ Gompertz Population Model

In the case of Malthus model the population grows exponentially and becomes ridiculously large. The exponential model finally becomes meaningless, since really the population never goes to infinity. In an attempt to construct a growth model more realistically we device the following Gompertz model : We assume a growth phenomena in which the growth coefficient is no longer constant, but vary with time t. Thus, in the case of exponential growth, we commence with the differential equation

$$\frac{dx}{dt} = rx \qquad \ldots \qquad (2.19)$$

where the growth coefficient r is assumed to change with time according to the relation

$$\frac{dr}{dt} = -\alpha r \qquad \ldots \qquad (2.20a)$$

where α is a decaying coefficient of r and we assume $\alpha > 0$. With the initial condition $r(o) = r_0$, the solution of (2.20a) is

$$r = r_0 e^{-\alpha t} \qquad \ldots \qquad (2.20b)$$

So the main feature of Gompertz growth model is the inclusion of an exponentially decreasing growth coefficient. Substituting (2.20b) in (2.19), we have

$$\frac{dx}{dt} = r_0 e^{-\alpha t} x \qquad \ldots \qquad (2.21)$$

The solution of (2.21) is

$$x(t) = x_0 \exp\left[\frac{r_0}{\alpha}\left(1 - e^{-\alpha t}\right)\right] \qquad \ldots \qquad (2.22)$$

with initial condition $x(o) = x_0$.

The equation (2.21) is the Gompertz equation and the function (2.22) is the Gompertz function. From (2.22) we see that as $t \to \infty$,

$$x \to x^* = x_0 \exp\left[\frac{r_0}{\alpha}\right] \qquad \ldots \qquad (2.23)$$

Substituting this value in (2.22), we have the alternative form of Gompertz function :

$$x(t) = x^* \exp\left[-\frac{r_0}{\alpha} e^{-\alpha t}\right] \qquad \ldots \qquad (2.24)$$

The quantity x^* is the value of x when t becomes very large, that is, it is the asymptotic value of x. In this sense it is the carrying capacity. Again using (2.24) in (2.21), we have the alternative form of Gompertz equation as

$$\frac{dx}{dt} = -\alpha x \log \frac{x}{x^*} \qquad \ldots \qquad (2.25)$$

We note that the specific growth rate $\frac{1}{x}\frac{dx}{dt}$ is given by the difference of the logarithms of x^* and x.

A comparison is made of exponential, logistic and Gompertz growth curves in the Fig. (2.6) below.

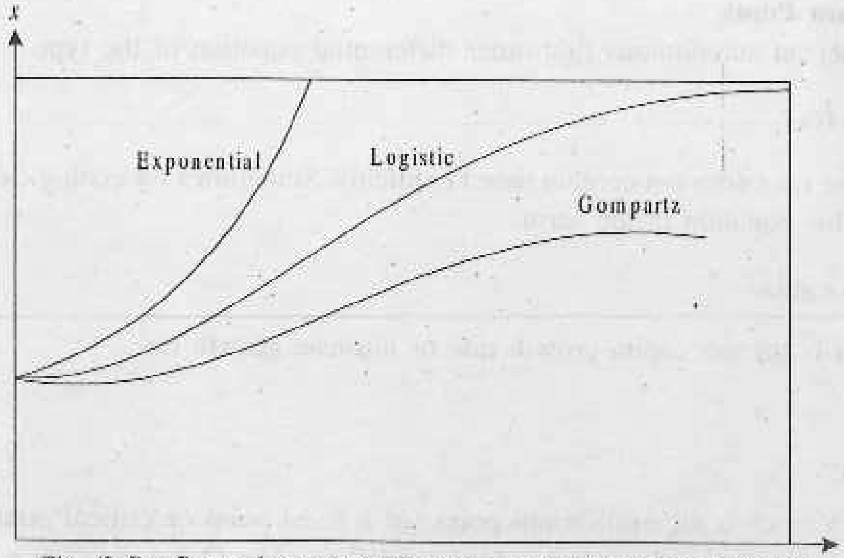


Fig. (2.6) : Comparison of Exponential, logistic and Gompertz curves.

2.7 ❑ Model Equations : Qualitative Analysis

Having the dynamical equations for model system in our hand, our next problem is to solve these equations. There are two approaches to it. First, we can attempt to find out an analytical solution, that is, a formula relating the value of the state variable at a time t to its value at some initial time t = 0 (say). When analytical solution is available, it provides a complete characterization of the dynamics of the given system. However, except for the simplest models, analytical solutions appear to be impossible. In the other case, an explicit solution can be calculated numerically. A numerical solution of differential equation is more tricky than that of difference equation. Numerical solutions are much less useful than analytical solutions, being valid only for chosen values of the initial state and model parameters. However, they are very easy to compute, and for simple system it is possible to obtain considerable insight by 'numerical experiment'. For more complicated models numerical solution is typically the approach available. In reality, vast majority of investigations proves that it is impossible to obtain complete or near complete information about a dynamical system, either by analytic solution or by numerical experiment. For this reason, over the last century or so mathematicians have developed methods or techniques of determining the qualitative properties of the solutions of the dynamical equations and thus answering many questions of ecological interests, without explicitly solving the model equations concerned.

Equilibrium Point:

We consider an autonomous first-order differential equation of the type

$$\frac{dx}{dt} = f(x) \quad \cdots \quad (2.26)$$

in which the r.h.s does not contain time t explicitly. Sometimes for ecological systems we write this equation in the form,

$$\frac{dx}{dt} = x\, g(x) \quad \cdots \quad (2.27)$$

where g(x) is the per capita growth rate or intrinsic growth rate.

Definition :

The point $x = x^*$ is an equilibrium point (or a fixed point or critical point or rest point or steady-state point) of the model equation (2.26) if $f(x^*) = 0$. If x(t) is a solution of the differential equation (2.26) that tends to a limit as $t \to \infty$, then it is not difficult to show that its limiting value must be equilibrium point. In fact, for a first-order differential equation every solution must either tend to an equilibrium point as $t \to \infty$ or be unbounded. However, not every equilibrium is a limit of non-constant solutions. For example, the only solution of the logistic equation that tends to zero as $t \to \infty$ is the identically zero solution.

Linearization :

In order to describe the behaviour of solution near equilibrium we introduce the process of linearization. If x^* be an equilibrium point of the equation (2.26) so that $f(x^*) = 0$, we make the change of variable $u(t) = x(t) - x^*$ representing the deviation of the solution from the equilibrium value. Putting this in equation (2.26) we have

$$\frac{du}{dt} = f(x^* + u(t)) + f(x^*)u(t) + \frac{f'(c)u(t^2)}{2}$$

where $x^* < c < x^* + u(t)$.

Since $f(x^*) = 0$, we have $\quad \frac{du}{dt} = f'(x^*)u(t) + h(u)$

where $h(u) = \frac{f'(c)u^2(t)}{2}$

For u (t) very small we can neglect h(u) so that we have the linear equation,

$$\frac{du}{dt} = f'(x^*)u \quad \ldots \quad (2.28)$$

The importance of linearization lies in the fact that the behaviour of its solution is easy to analyse and this behaviour also describes the behaviour of the solution of the original equation (2.26) near equilibrium. We have, infact, the theorem :

Theorem 2.1 :

If all solutions of the linearization (2.28) at equilibrium x^* tends to zero as $t \to \infty$, then all solutions of (2.26) with the initial point x(o) sufficiently close to x^*, tends to the equilibrium point x^* as $t \to \infty$.

Stability :

The process of linearization plays an important role in the study of the stability of the equilibrium point or state. For this, let us first give a formal definition of stability.

Definition :

An equilibrium point x^* is Lyapunov stable if for any arbitrary small $\varepsilon > 0$, there exists a $\delta > 0$ (depending on ε) such that, for all initial condition $x(o) = x_0$ satisfying $|x_0 - x^*| < \delta$, we have $|x(t) - x^*| < \varepsilon$ for all $t > 0$. In other words, an equilibrium point is stable if starting close (enough) to equilibrium guarantees that you will stay close to equilibrium. An equilibrium point x^* is asymptotically stable if it is stable and if in addition $|x_0 - x^*| < \delta$ implies $\lim_{t\to\infty} x(t) = x^*$

Remark :

In biological applications, we will ordinarily require asymptotic stability rather than stability. This is because asymptotic stability can be determined form the linearization, while stability cannot and again this is because asymptotic stable equilibrium is not disturbed greatly by a perturbation of the differential equation. In term of asymptotic stability we may restate the theorem (2.1) as follows :

Theorem 2.2 :

An equilibrium point x^* of (2.26) is asymptotically stable, if $f'(x^*)<0$ and unstable if $f'(x^*)>0$.

Exercises

(1) Investigate the asymptotic stability of the equilibrium points of the following model equations.

(i) $\frac{dx}{dt} = r\,x\left(1 - \frac{x}{k}\right)$ [Logistic model]

(ii) $\frac{dx}{dt} = -r\,x \log\frac{x}{k}$ [Gompertz model]

(iii) $\frac{dx}{dt} = \frac{r\,x\,(k - x}{k + a\,x}$ [Smith model]

(2) Show that if $r < 0$, $k < 0$, every solution of the logistic equation with $x(o) \geq 0$ approaches zero as $t \to \infty$.

(3) A population is governed by the equation

$$\frac{dx}{dt} = x\,(e^{3-x} - 1)$$

Find all equilibria and determine their stability.

(4) Discuss the model

$$\frac{dx}{dt} = r\,x\left(1 - \frac{x}{k}\right)\left(\frac{x}{k_o} - 1\right)$$

where $0 < ko < k$. Find all limits of solutions with $x(o) > 0$ as $t \to \infty$ and find the set of initial values corresponding to each limit.

(5) Show that for every choice of the constant c, the function

$$x = \frac{k}{1 + c\,e^{-rt}}$$

is a solution of the logistic differential equation.

(6) Consider the logistic equation

$$\frac{dx}{dt} = r\,(t)\left(1 - \frac{x}{k}\right), \quad x(o) = x_o$$

with time—dependent intrinsic growth rate r(t). Show that the solution is given by

$$x(t) = \frac{kx_0}{x_o + (k - x_o)e^{-\int_0^t r(s)ds}}$$

2.8 ❑ Harvest Models :

We wish to study the effect on a population model of the removal of members of the population at a specified rate. If a population modeled by the differential equation

$$\frac{dx}{dt} = f(x) \qquad \cdots \qquad (2.29)$$

is subjected to a harvest at a rate h(t) member per unit time for some given function h(t), then the harvested population is modeled by the differential equation,

$$\frac{dx}{dt} = f(x) - h(t) \qquad \cdots \qquad (2.30)$$

If the numbers are removed at a constant rate H (constant) per unit time, then the model equation is

$$\frac{dx}{dt} = f(x) - H \qquad \cdots \qquad (2.31)$$

Such type of harvesting is called constant rate or constant yield harvesting. It arises when a quota is specified (for example, through permit as in deer hunting seasons in many states or by agreement as sometimes in whaling). If the population is governed by logistic equation, then the harvest model equation is,

$$\frac{dx}{dt} = r\,x\left(1 - \frac{x}{k}\right) - H \qquad \cdots \qquad (2.32)$$

The equilibrium points are given by

$$r\,x\left(1 - \frac{x}{k}\right) - H = 0 \text{ or } x^2 - kx + \frac{kH}{r} = 0$$

$$\text{or} \quad x_1 = \frac{1}{2}\left\{k - \sqrt{k^2 - \frac{4Hk}{r}}\right\} \text{ and } x_2 = \frac{1}{2}\left\{k + \sqrt{k^2 - \frac{4Hk}{r}}\right\} \cdots \quad (2.33)$$

provided $k^2 - \frac{4kH}{r} \geq 0$ or $h \leq \frac{rk}{4}$. If $H > \frac{rk}{4}$ both roots are complex, $x'(t) < 0$ for all x, and every solution crashes, hitting zero in finite time. If a solution reaches zero in finite time, we consider system to have collapsed. If $0 \leq H < \frac{rk}{4}$, there are two equilibria : x_1 which increases from 0 to k/2 as H increases from 0 to rk/H and x_2, which decreases from k to k/2 as H increases. The stability of an equilibrium x* of $\dot{x} = F(x) - H$ requires $F^1(x^*) < 0$, which for logistic model means $x^* > k/2$. Thus x_1 is always unstable and x_2 is always asymptotically stable. When H increases to the critical value $H_c = rk/4$, there is a discontinuity in the behaviour of the system - the two equilibria coalesce and annihilate each other. For $H < H_c$ the population size tends to an equilibrium size that approaches k/2 as $H \to H_c$ (provided the initial population size is at least x_1), but for $H > H_c$ the population size reaches zero in finite time for all initial populations sizes (see Fig. (2.8a) below). Such a discontinuity is called a (mathematical) Catastrope; the biological implications are catastropic to species being modeled.

For a general model $x' = f(x) - H$ equilibria are given by $f(x) - H = 0$, that is, by finding values x^* of x for which the growth curve $y = f(x)$ and the harvest curve $y = H$ (a horizontal line) intersect. An equilibrium x^* is asymptotically stable if $(f(x) - H)'|_{x=x^*} = f'(x^*) < 0$, that is, if at such an intersection the growth curve crosses the harvest curve from above to below as x in increases (see fig. (2.8b)). From fig (2.8b) it is clear that if $H > \max f(x)$ there is no equilibrium, and the critical harvest rate H_c at which two equilibria coalesce and disappear is $\max f(x)$.

There are other models of harvesting for example, the harvest rate h(t) may be a linear function of population size : $h(t) = Ex$ and in that case it is known as constant effort harvesting. Harvesting plays an important role in fishery and forestry and has economic and commercial importances.

2.9 ❑ Summary

(i) We have first stated basic biological and mathematical postulates necessary for the development for the continuous-time models of populations,

(ii) We have set up a general model equation for single-species populations. We have studied some basic single-species population growth models, namely Malthus growth model,Logistic growth model, Allee effect, Gompertz growth model and Harvest model etc.

(iii) For the qualitative analysis of the model equations we have discussed a autonomous first order differential equations, its equilirium (or fixed) points and criteria of local stability of equilibrium points.

(iv) As illustrative examples, we have discussed some problems related to the model equations.

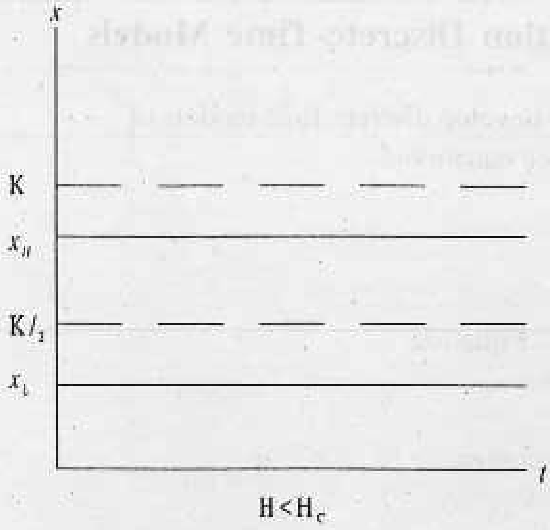


Fig. (2.8a)

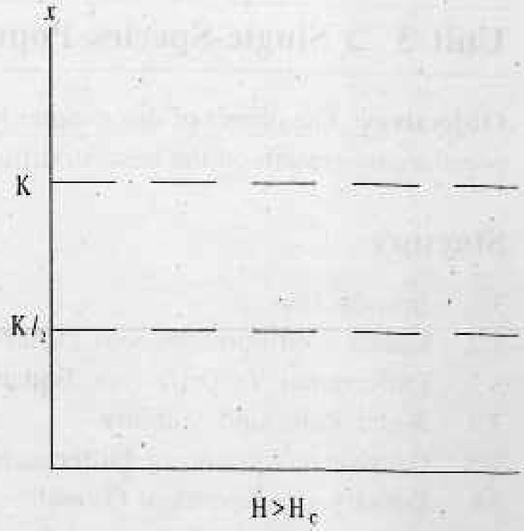


Fig. (2.8b)

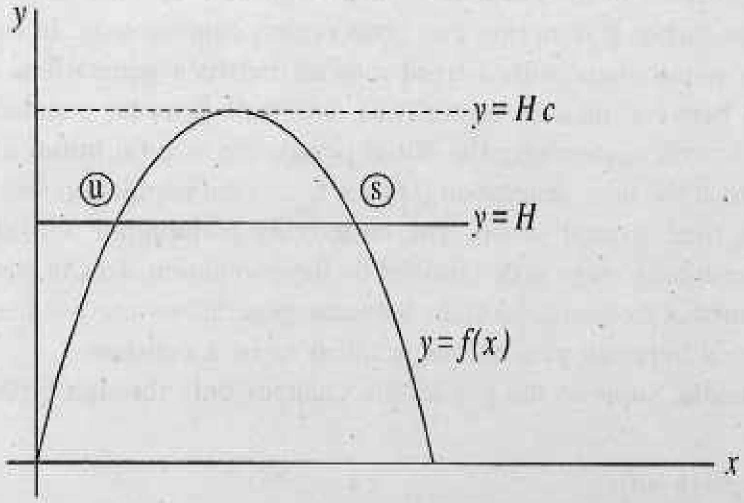


FIGURE : 2.8b Intersections of the growth curve with the line of constant yield

Unit 3 ❑ Single-Species Population Discrete-Time Models

Objectives: The object of this chapter is to develop discrete-time models of populations growth on the basis of difference equations.

Stucture

3.1 ❑ Introduction

For many organisms, births occur in regular, well-defined 'breeding seasons'. This contradicts our earlier assumption that birth occurs continuously. In this chapter we shall consider populations with a fixed interval between generations or possibly a fixed interval between measurement. Thus, we shall describe population size by a sequence $\{x_n\}$, with x_0 denoting the initial population size (at initial time t_0), x_1 the population size at the next generation (at time t_1), x_2 the population size at the second generation (at time t_2) and so on. The underlying assumption will always be that population size at each stage is determined by the popuation sizes in past generations; but that intermediate population sizes between generations are not needed. Usually the time interval between generations is taken to be a constant.

For example, suppose the population changes only through births and deaths, so that

$$x_{n+1} - x_n = (b - d)x_n \quad \cdots \quad (3.1)$$

where bx_n is the number of births and dx_n is the number of deaths in the time-interval (t_n, t_{n+1}); b and d (assumed constants) are the birth and death rates respectively. From (3.1) we thus have

$$x_{n+1} = (1 + b - d)x_n = rx_n, x(t_0) = x_0 \quad \cdots \quad (3.2)$$

which is a first-order linear homogeneous difference equation. The growth rate $r = 1 + b - d$ is a parameter of the equation. In general, we can write a first-order difference equation as

$$x_{n+1} = f(x_n), \text{ with } x(t_0) = x_0 \quad \ldots \quad (3.3)$$

In such an equation the new value of x is determined completely by the previous value. In higher-order difference equation we would require information about several previous values to determine the current value. For example, the Kepler recursion relation

$$x_{n+1} = x_n + x_{n-1} \quad \ldots \quad (3.4)$$

is a second-order difference equation as it requires two previous values x_n and x_{n-1} to find out the exact value of x_{n+1}. Such type of difference equation appears in the case of over lapping generations as in the case of Snow Geese in Baffin Island. The function f in (3.3) is called a map or iteration. A map f is linear if f is of the form $f(x) = ax$, for some constant a Otherwise the equation (3.3) is non-linear (or density — dependent in biology).

Example (3.1) : Logistic Difference Equation

Let x_n be the size of a population of a certain species at time t_n. Let r be the rate of growth of population from generation to generation. Then from (3.2) we have

$$x_{n+1} = rx_n, \qquad r > 0 \quad \ldots \quad (3.2)$$

with initial population $x(to) = x_0$.
Then by simple iteration we find that

$$x_n = r^n x_0 \quad \ldots \quad (3.5)$$

is the solution of (3.2). If $r > 1$, the population increases without any bound to infinity. If $r = 1$, $x_n = x_0$, the population stays constant forever. If $r < 1, \lim_{n\to\infty} x_n = 0$, the populations eventually becomes extinct.

We observe that for most species the above model is not realistic, the population increases until it reaches a threshold. Then limited resources would force the members of the species to fight and compete with others. This competition is proportional to the number of squabbles x^2_n among them. A more realistic model is, therefore,

$$x_{n+1} = rx_n - bx^2_n \quad \ldots \quad (3.6)$$

where b is the proportionality constant of interaction among the members of the species. Writing $y_n = \frac{b}{r} x_n$, we hvae

$$y_{n=1} = ry_n(1 - y_n) \quad \ldots \quad (3.7)$$

The equation (3.7) is the discrete logistic equation and the map $f(y) = ry(1 - y)$ is called the logistic map. It is a reasonably good model in which generations do not overlap. The logistic equation (3.7) is very important, by varying the value of the parameter r, this simple and innocent looking equation exhibits somewhat complex behaviours.

3.2 ❑ Linear Non-homogeneous Difference Equation

Consider the first-order linear non-homogeneous difference equation

$$x_{n+1} = a\,x_n + b, \qquad x(t_0) = x_0 \qquad \ldots \quad (3.8)$$

The equation can be solved by successive iterations,

$$x_1 = ax_0 + b$$

$$x_2 = ax_1 + b$$

$$= a(ax_0 b) + b = a^2x_0 + ab + b$$

By induction, we can show that

$$x_3 = ax_2 + b = a(a^2x_0 + ab + b) + b$$

$$= a^3x_0 + a^2b + ab + b$$

By induction,we can show that

$$x_n = a^n x_0 + \sum_{j=0}^{n-1} a^{n-j-1}b$$

$$= a^n x_0 + b\left(\frac{a^n - 1}{a - 1}\right), \text{ if } a \neq 1 \qquad \ldots \quad (3.9)$$

The above formula (3.9) is an important result having many applications. As an application let us consider the following problem.

Example (3.2) :

A drug is administered every six hours. Let D(n) be the amount of the drug in the blood system at the nth interval. The body eliminates a certain fraction p of the drug during each time interval. If the initial blood administered is D_0, find D(n) and $\lim_{n \to \infty} D(n)$.

Solution :

The first step is to write down the difference equation that relates the amount of drug in the patient blood system D(n+1) at the time interval (n + 1) with D(n). The resulting equation is

$$D(n) = (1-p)\,D(n) + D_o,$$

Using the formula (3.9). We have

$$D(n) = (1-p)^n D_o + D_o\left[\frac{1-(1-p)^n}{p}\right]$$

$$= \left[D_o - \frac{D_o}{p}\right](1-p)^n + \frac{D_o}{p}$$

$$\text{Thus } \lim_{n\to\infty} D(n) = \frac{D_o}{p}$$

3.3 ❑ Differential Vs Difference Equation

Consider the differential equation

$$\frac{dx}{dt} = g(x(t)), \qquad x(t_o) = x_o \qquad \dots \qquad (3.10)$$

For many differential equations such as (3.10), it may not be possible to find a 'closed form' of solution. In that case, we resort to numerical method to approximate the solution of (3.10). In Euler algorithm, for example, we start with a discrete set of points $(t_0, t_1, ..t_n)$ with $h = t_{n+1} - t_n$ as the step size. Then for $t_n < t < t_{n+1}$ we approximate $x(t)$ by $x(t_n)$ and dx/dt by

$$\frac{dx}{dt} = \frac{x(t_{n+1}) - x(t_n)}{h}$$

The equation (3.10) then leads to the equation

$$x(t_{n+1}) = x(t_n) + h\,g(x(t_n))$$

or in simple form

$$x_{n+1} = x_n + h\,g(x_n) \qquad \dots \qquad (3.11)$$

where $x_n = x(t_n)$

The equation (3.11) is of the form (3.3) with

$$f(x) = x + h\,g(x) \qquad \dots \qquad (3.12)$$

Given initial data $x(t_0) = x_0$, we may use the equation (3.11) to generate the values $x(t_1), x(t_2).....x(t_n)$. These values approximate the solution of the differential equation (3.10) at the grid points $t_1, t_2........t_n$ provided that h is sufficiently small.

3.4 ❑ Fixed (or Equilibrium) Point and Stability

When the map f is linear it is possible to obtain 'closed form' of solution of the first-order difference equation (3.3). However, the situation changes drastically when the map f is non-linear. Since we can not solve all the non-linear difference equations, it is important to develop qualitative or graphical method of finding the behaviour of the solutions. Of particular importance is the finding the fixed points or equilibrium points or steady-states.

Definition : A point x^* is said to be a fixed point or an equilibrium point of the difference equation $x_{n+1} = f(x_n)$ if $f(x^*) = x^*$.

One of the objectives in the theory of dynamical system is the study of the behaviour of the system, that is, the behaviour of solutions of a difference equation near the fixed or equilibrium point. Such a program of investigation is called stability theory. Let us now explain the concept of stability of a fixed point.

Definition : Let x^* be a fixed (or equilibrium) point of the difference equation

$$x_{n+1} = f(x_n), \qquad x(t_o) = x_o.$$

Then

(i) x^* is said to be stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|x_0 - x^*| < \delta$ implies $|x_n - x^*| < \varepsilon$ for all positive integers n and for all x in the domain of definition. Otherwise the point x^* will be unstable.

(ii) x^* is said to be attracting if there exists $\eta > 0$ such that $|x_0 - x^*| < \eta$ implies $\lim_{n \to \infty} x_n = x$

(iii) x^* is asymptotically stable (sometimes called a sink) if it is both stable and attractive. If in (ii) $\eta = \infty$, then x^* is said to be globally asymptotic stable.

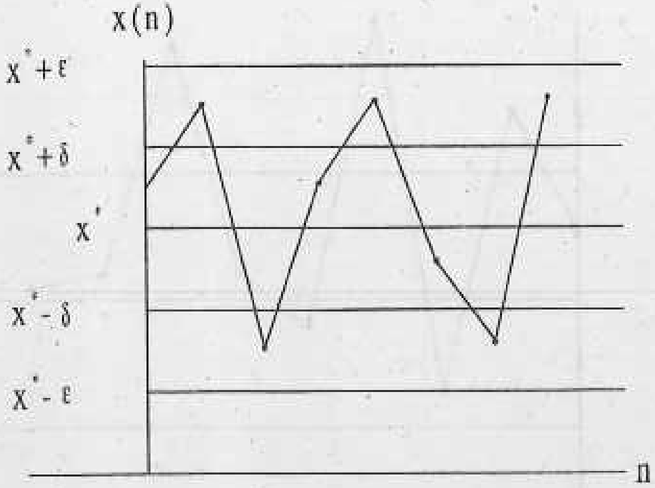


Fig. 3.1 : Stable fixed point x*

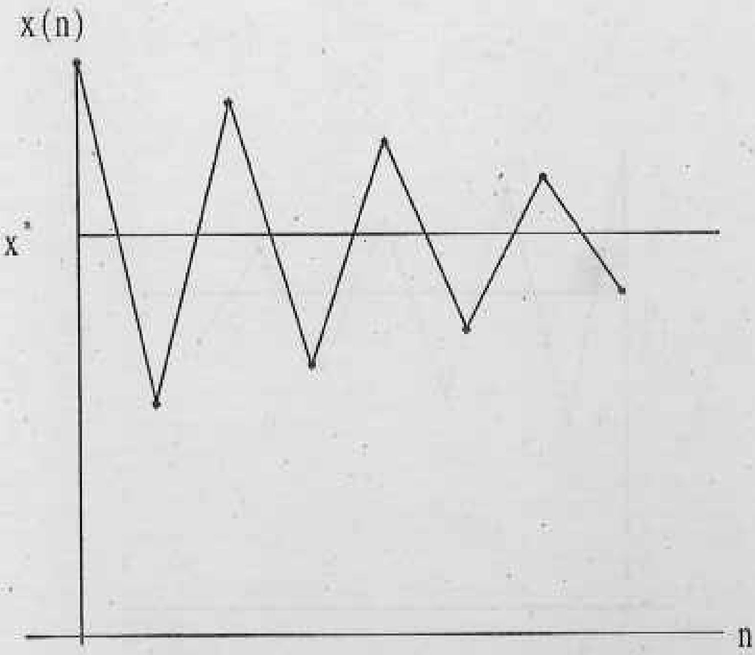


Fig. 3.2 : Unstable fixed point

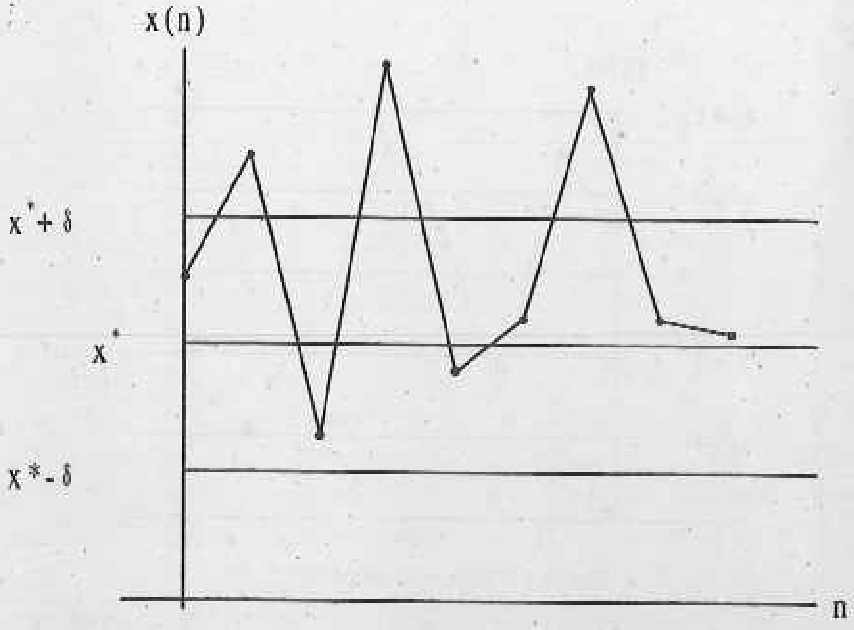


Fig. 3.3 : Attractive, but unstable fixed point x

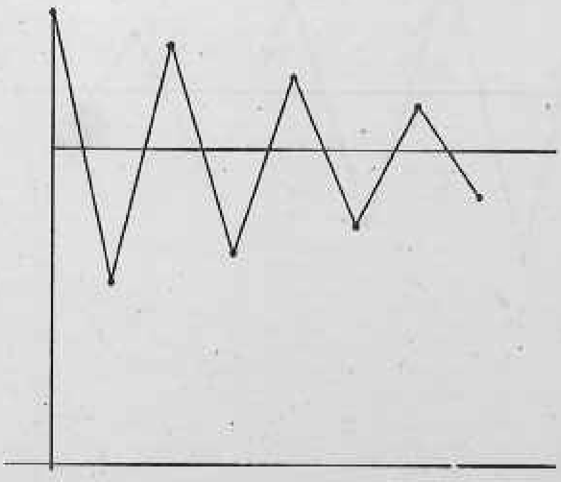


Fig. 3.4 : Asymptotically stable fixed point x

3.5 ❑ Graphical Solution of Difference Equation

In example (3.2) we have explained the method of solving a first-order linear difference equation. Let us now describe a graphical method of solving difference equation of the form $x_{n+1} = f(x_n)$ by a graphical method; called "Cobweb diagram". It is also one of the effective graphical iteration methods to determine the stability of fixed point.

Cobweb Diagram :

We start with an initial point x_o. Then we move vertically until we hit the graph $y = f(x)$ at the point $(x_0, f(x_0))$. We then travel horizontally to meet the line $y = x$ at the point $(f(x_o), f(x_0))$. This determines $f(x_0) = x_1$ on the x-axis. To find out the next iterated value $x_2 = f(x_1) = f(f(x_o)) = f^2(x_o)$, we move again vertically until we strike the graph $y = f(x)$ at the point $(f(x_0), f^2(x_o))$; and then move horizontally to meet the line $y = x$ at the point $(f^2(x_o), f^2(x_o))$. This determines $x_2 = f^2(x_o)$ on the x-axis. Proceeding in this way, we can evaluate all of the iterated values $\{x_1, x_2, \ldots, x_n \ldots\}$. Let us explain this method with a simple example and show how it can be used to test the stability of a fixed point. Note that fixed point is the point of intersection of the curve $y = f(x)$ and the line $y = x$.

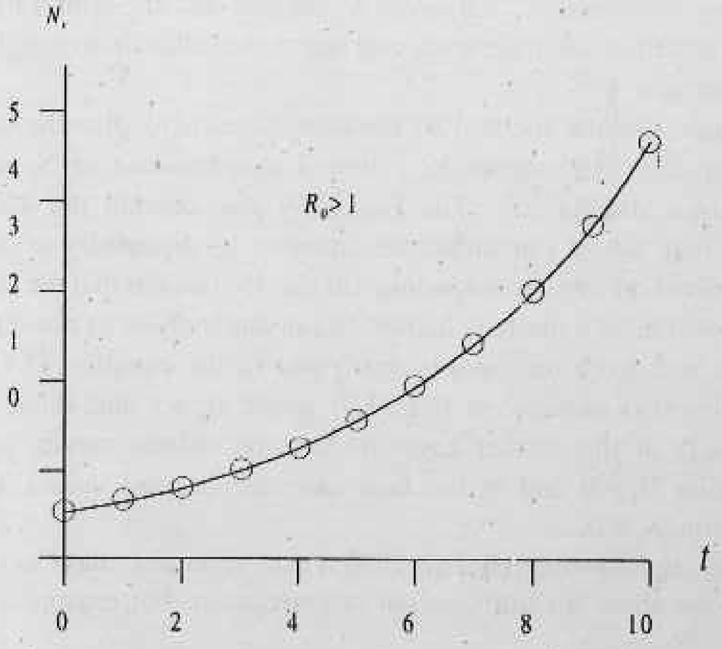


Fig. : 3.5 Geometric growth.

3.6 ❑ Density — Independent Growth

Let N_t be the size of the population in year (or generation) t. We will census the population each year at the same stage of the life cycle. Imagine that each individual leaves Ro offsprings before dying. We shall call Ro the net reproductive rate. It follows that

$$N_{t+1} = R_0 N_t \quad \ldots \quad (3.13)$$

The equation (3.13) is linear, first-order, constant - coefficient difference equation. The solution is given by (see 3.5)

$$N_t = R^t_0 N_0 \quad \ldots \quad (3.14)$$

The solution is thus one of geometric growth or decay. If Ro > 1, each individual leaves more than one descendant, and the population grows geometrically (see Fig. 3.5).

If $0 < R_0 < 1$ the individuals leave, on average, fewer than one descendant, and the population declines geometrically (see Fig. 3.6). These figures resemble those for exponential growth and decay. Individuals cannot leave a negative number of offspring. However, nothing can prevent us from pondering this possibility mathematically. For $-1 < R_0 < 0$, we get decaying oscillations (Fig. 3.7). For $R_0 < -1$, we get growing oscillation (Fig. 3.8). Figures (3.5) to (3.8) suggest that the solutions approach the origin if Ro is less than 1 in magnitude and that these solution diverge if Ro is greater than 1 in magnitude.

Let us now use the method of cobweb diagram to plot the solution of the equation (3.13). Fig. (3.9) shows N_{t+1} plotted as a function of N_t for $R_0 > 1$. The curve is clearly a straight line. The Fig. (3.9) also contain the 45° dashed line, $N_{t+1} = N_t$. We may iterate our difference equation by repeatedly (a) moving up (or down) to the curve and then (b) bounching off the 45° line (so that we reset $N_{t+1} = N_t$). This approach will be of extremely helpful late in our analysis of non-linear difference equation. Zero is a fixed (or equilibrium) point of the equation (3.13). The trivial equilibrium $N_t = 0$ is unstable in Fig. (3.9) where $R_0 > 1$ and stable in Fig. (3.10) where $0 < R_0 < 1$; in the former case the iterated values moves away from the equilibrium point $N_t = 0$ and in the later case the iterated values approaches the equilibrium point $N_t = 0$.

A linear, density-independent difference equation may have a non-zero equilibrium if we allow for immigration or emigration. For example, the difference equation

$$N_{t+1} = \frac{3}{4} N_t + 10 \quad \ldots \quad (3.15)$$

has an equilibrium $N^* = 40$.

We now introduce a new variable $x_t = N_t - 40$

Then the equation (3.15) becomes $x_{t+1} = \frac{3}{4} x_t$

So that $x_t = x_0 \left(\frac{3}{4}\right)^t$

$$\text{or,} \quad N_t = 40 + (N_0 - 40)\left(\frac{3}{4}\right)^t \quad \ldots \quad (3.16)$$

The small perturbation about the equilibrium decay; the equilibrium is asymptotically stable. This stability also comes out in Cobweb analysis [see Fig.(3.9) (3.12)].

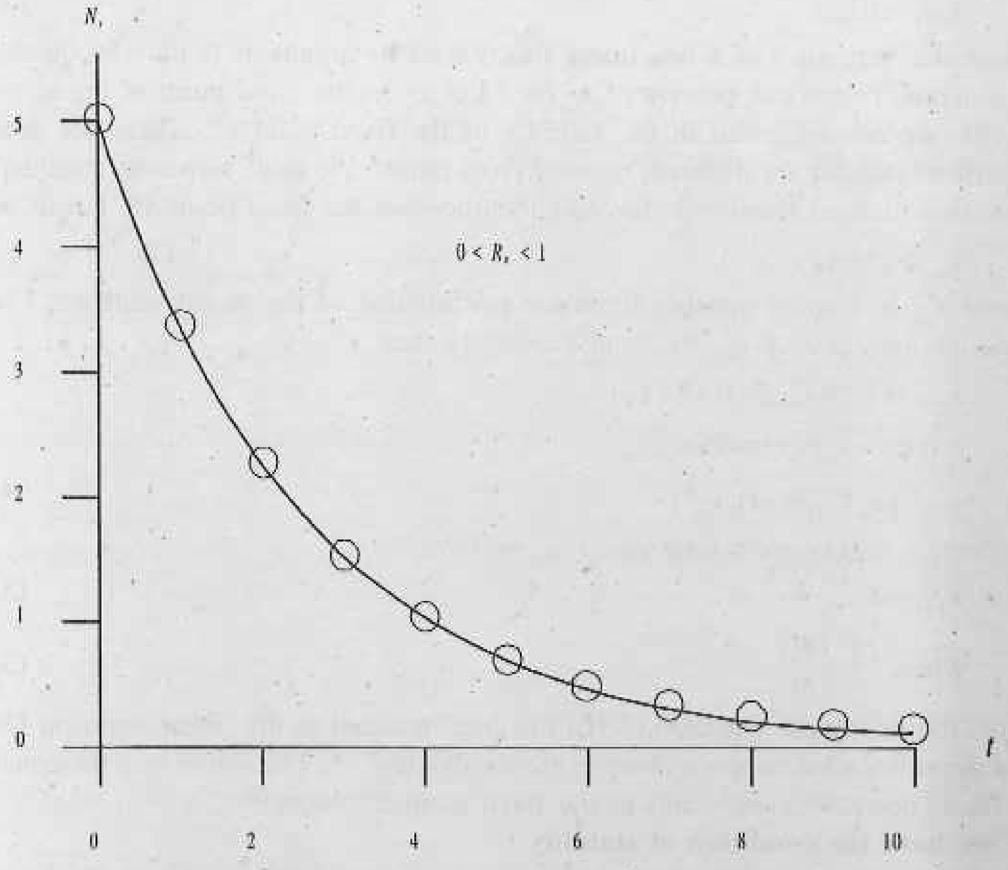


Fig. : 3.6 Geometric decay.

3.7 ❑ Steady-state : Criteria of Stability

The stability is of fundamental importance in biology. When a steady-state (or an equilibrium state) is unstable, great changes may about to happen : a population may crash, homeostasis may be disrupted or else the balance in number of competing groups may shift in favour of a few. Thus, even if an exact analytical solution is not easy to come by, qualitative information about whether change is imminent is of potential importance.

Let us now find out the criteria of stability of a fixed equilibrium point or a steady-state. We consider the non-linear first-order difference equation of the form (3.3) :

$$x_{n+1} = f(x_n) \quad \ldots \quad (3.16)$$

where the function f is a non-linear function of its argument (f may be quadratic, exponential reciprocal, powers of x_n etc.) Let x^* be the fixed point of the equation (3.16). We are interested in the stability of the fixed point x^*. There are general criteria of stability for different types of fixed points. We shall, however, consider the condition of local stability in the neighbourhood of the fixed point x^*. Let us write

$$x_n = x^* + x'_n \quad \ldots \quad (3.17)$$

where x'_n is a small quantity termed a perturbation of the steady state x^*. Let us linearise the equation (3.16) about the steady state x^*

$$x_{n+1} = x^* + x'_{n+1} = f(x^* + x'_n)$$

$$= f(x^*) + x'_n f'(x^*) + O(x'^2_n)$$

$$= x^* + x'_n f'(x^*) + O(x'^2_n)$$

Neglecting higher-order term $O(x_n'^2)$, we have

$$x'_{n+1} = a\, x'_n \quad \ldots \quad (3.18)$$

$$\text{where} \quad a = \left.\frac{df}{dx}\right|_{x=x^*} \quad \ldots \quad (3.19)$$

Thus, the non-linear equation (3.16) has been reduced to the linear equation (3.18) that describes what happens close to the steady-state x^*. The solution of the equation (3.18) is decreasing and tends to the fixed point x^* wherever $|a| < 1$.

So we have the condition of stability :

The fixed point or steady-state x^* is asymptotically stable if and only if

$$|f'(x^*)| < 1 \quad \ldots \quad (3.20)$$

Note that whenever $|f'(x^*)| = 1$, then $(x'_{n+1} - x'_n)$ implying a constant deviation is unable to decrease x'_n to reach the fixed point. More formally, we have the theorem.

Theorem :

Let x^* be a fixed point of the difference equation (3.16). Suppose that $f(x)$ is continuously differentiable and $|f'(x^*)| \neq 1$. Then, the fixed point x^* is asymptotically stable if $|f'(x^*)| < 1$ and unstable if $|f'(x^*)| > 1$.

Example (3.3) :

The growth of a population satisfies the following difference equation

$$x_{n+1} = \frac{kx_n}{b+x_n}, \; b,k > 0,$$

Find the steady-state (if any). If so, is that stable ?

Solution : Let x^* be the steady-state value of x_n.

Then $\qquad x^* = x_{n+1} = x_n$

So that $\qquad x^* = \dfrac{kx^*}{b+x^*}, \quad$ or $\quad x^* = 0, k-b$

The steady-state makes sense only if $x^* > 0$ i.e., if $k > b$, since negative population is biologically meaningless. To study the stability, we consider the equation

$$x'_{n+1} = a\,x'_n$$

where $\quad a = \left.\dfrac{df}{dx}\right|_{x^*} = \left.\dfrac{d}{dx}\left(\dfrac{kx}{b+x}\right)\right|_{x^*=k-b} = \dfrac{b}{k}$

So the steady-state $x^* = k - b$ is stable if $\left|\dfrac{b}{k}\right| < 1$. Since both k and b are positive, the condition of stability reduces to $k > b$. The study of stability of $x^* = 0$ is left as an exercise.

Excercise :

(1) Find the non-negative equilibrium of a population governed by

$$x_{n+1} = \frac{2x_n^2}{x_n^2+2}$$

and check the stability.

(2) A population is governed by the equation

$$x_{n+1} = x_n e^{3-x_n}$$

show that all equilibria are unstable.

Example (3.4) :

Logistic Difference Equation Revisited

Consider the following equation

$$x_{n+1} = r\,x_n(1-x_n).$$

Investigate the stability of the steady-state.

Solution : Let x* be the steady-state value of x. Then

Then $\qquad x^* = r\,x^*(1-x^*)$

So that $\qquad x_1^* = 0, \qquad x_2^* = 1-\frac{1}{r}.$

Perturbation about x_2^* satisfies.

$$x'_{n+1} = a\,x'_n$$

where $\quad a = \left.\frac{df}{dx}\right|_{x_2^*} = r(1-2x|_{x_2^*=1-\frac{1}{r}} = 2-r$

Thus, $x_2^* = 1-\frac{1}{r}$ is stable whenever $|a|<1$ which implies $|2-r|<1$ or $1<r<3$. Then the stability of the steady-state $x^* = 1-\frac{1}{r}$ is conditioned on a parameter r. If r is greater or small than certain critical values (here 3 and 1); the steady-state $x_2^* = 1-\frac{1}{r}$ is not stable. Such critical parameter values, often called <u>bifurcation values</u>; are points of demarcation for abrupt behaviour of the equation of the system it models. There may be multitude of such transitions, so that as increasing values of the parameter are used, one encounters different behaviours. In fact, if we increase the value of r beyond 3 the equation will exhibit the complex behaviour of period doubling and chaos. This is, however, beyond our discussion.

Example (3.5) :

Density-Dependent Growth

An assumption that growth rate depends on the density of the population leads us to consider models of the following form :

$$N_{t+1} = f(N_t)$$

where $f(N_t)$ is some non-linear function of the population density. Quite often, single-species populations (insects, for example) are described by such equation. We consider the following model.

Let the single-species population satisfies the equation

$$N_{t+1} = \left(\frac{\lambda}{\alpha}\right) N_t^{1-b}$$

where λ is the reproductive rate, assumed to be greater than 1. The equation is best understood in the form

$$N_{t+1} = \left(\frac{1}{\alpha} N_t^{-b}\right)(\lambda\ N_t)$$

→ No of progeny at generation t.

→ Fraction that survive to generation (t+1).

where α, b, $\lambda > 0$. Since the fraction of survivors can not exceed 1, we see that the population must exceed a certain size N_c for this model to be biologically applicable. The steady-state population size is given by

$$N^* = \frac{\lambda}{\alpha} N^{*1-b} \quad \text{or} \quad N^* = \left(\frac{\lambda}{\alpha}\right)^{\frac{1}{b}}$$

We write $f(N) = \left(\frac{\lambda}{\alpha}\right) N^{1-b}$ then $\left.\frac{df}{dN}\right|_{N=N^*} = 1-b$ so the stability of the steady-state

$N^* = \left(\frac{\lambda}{\alpha}\right)^{\frac{1}{b}}$ requires $|1-b| < 1$ or $0 < b < 2$.

We note that b = 0 is a situation in which the survivorship is not density dependent; then the population grows at the rate $\left(\frac{\lambda}{\alpha}\right)$. Thus, lower-bound for stabilizing values of b makes sense.

Excercise :

(3) : Investigate the stability of the steady-state of the model equation

$$N_{t+1} = N_t \exp\left\{r\left(1-\frac{N_t}{k}\right)\right\}$$

(4) : Consider the tent map f :

$$f(x) = \begin{cases} 2x \text{ for } 0 \le x \le ½ \\ 2(1-x), \text{ for } ¼ x \le 1 \end{cases}$$

Write down the difference equation corresponding to the map f defined above. Find out the fixed points, draw the Cobweb diagram and investigate the stability.
The fixed points are obtained by the equations

$$2x = x \quad \text{and} \quad 2(1-x) = x$$

so the fixed points are

$$x^*_1 = 0 \quad \text{and} \quad x^*_2 = \frac{2}{3}$$

We observe from the Cobweb diagram (Fig. 3.12) that both the fixed points are unstable.

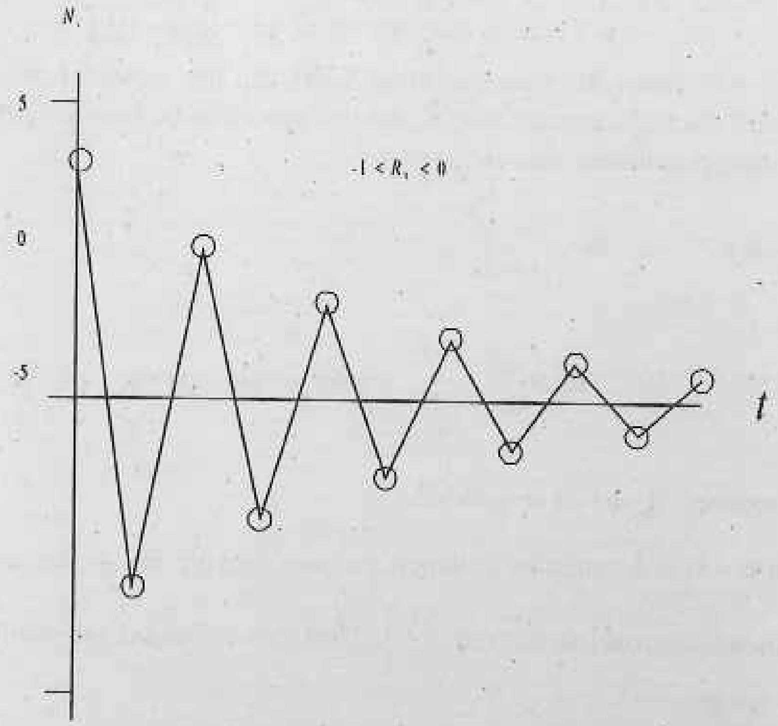


Fig. : 3.7 Decay oscillations.

3.8 ❑ Second-Order Difference Equation and Application

Let us now consider a second-order difference equation of the form

$$x_{n+1} = f(x_n, x_{n-1}) \quad \ldots \quad (3.20)$$

where the function f is now a function of two immediate preceding values x_n and x_{n-1}. The function f may be linear or non-linear. We shall, however, be confined to linear functions only. For simplicity, we shall consider a linear, homogeneous second-order difference equation of the form :

$$a_0 x_{n+1} + a_1 x_n + a_2 x_{n-1} = 0 \quad \ldots \quad (3.21)$$

To solve the equation (3.21) let us take a solution of the form : $x_n = C\lambda^n$. Putting this value in equation (3.21), we have,

$$a_0\lambda^2 + a_1\lambda + a_2 = 0 \quad \ldots \quad (3.22)$$

which is known as the characteristic equation of the equation (3.21). The general solution of the equation (3.21) is a linear superposition of the basic solutions of the equation. Let λ_1 and λ_2 be the two solutions of the characteristic equation i.e. the two eigenvalues. Then the generals solution of the difference equation is given by

$$x_n = A_1\lambda_1^n + A_2\lambda_2^n \quad \ldots \quad (3.23)$$

where A_1 and A_2 are two constants to be determined from two initial values of x. If the eigenvalues λ_1 and λ_2 are complex conjugates, we can transform the solution in polar-coordinates. Let $\lambda_1, \lambda_2 = a \pm i\,b$ and write $a = r\cos\phi$, $b = r\sin\phi$. So that

$$r^2 = (a^2 + b^2), \quad \phi = \tan^{-1}\frac{b}{a}.$$

Then $a + i\,b = r(\cos\phi + \sin\phi) = re^{i\phi}$.

$a + i\,b = r(\cos\phi - \sin\phi) = re^{i\phi}$.

So the general solution is

$$x_n = A_1(a + i\,b)^n + A_2(a - ib)^n \quad \ldots \quad (3.24)$$

$$= A_1 r^n (\cos n\phi + i\sin n\phi) + A_2 r^n(\cos n\phi - i\,\mathrm{Sin}\,n\phi)$$

$$= B_1 r^n \cos n\phi + i B_2 r^n \mathrm{Sin}\, n\phi$$

where B_1 and B_2 are two constants

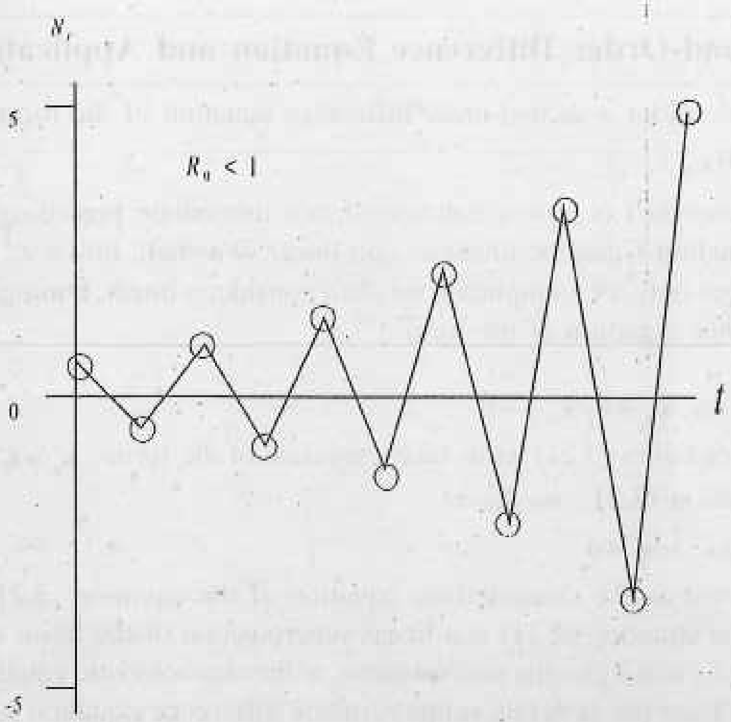


Fig. : 3.8 Growing oscillations.

3.9 Rabbit Problem : Fibonacci Sequence

Let us consider the linear second-order homogeneous difference equation

$$x_{n+1} = x_n + x_{n-1} \qquad \cdots \qquad (3.25)$$

which stems from Fibonacci work and the sequence $\{x_n\}$ is known as Fibonacci sequence. Let us put $x_n = c\lambda^n$ in (3.25) to have the characteristic equation

$$\lambda^2 = \lambda + 1 \qquad \cdots \qquad (3.26)$$

so that the eigenvalues are $\lambda_1 = \frac{1-\sqrt{5}}{2}$, $\lambda_2 = \frac{1+\sqrt{5}}{2}$ The general solution of the equation (3.25) is then given by

$$x_n = A_1\lambda_1^n + A_2\lambda_2^n = A_1\left(\frac{1-\sqrt{5}}{2}\right)^n + A_2\left(\frac{1+\sqrt{5}}{2}\right)^n$$

Suppose we start write $x_o = 0$, $x_1 = 1$ (initial condition).

So, $\quad 0 = A_1\lambda_1^o + A_2\lambda_2^o = A_1 + A_2$

$$1 = A_1\lambda_1 + A_2\lambda_2 = A_1\left(\frac{1-\sqrt{5}}{2}\right) + A_2\left(\frac{1+\sqrt{5}}{2}\right)$$

These give $A_1 = -\frac{1}{\sqrt{5}}, \quad A_2 = \frac{1}{\sqrt{5}}$

Then the general solution is $x_n = -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n + \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$ ... (3.27)

We observe that $\lambda_2 > 1, \quad -1 < \lambda_1 < 0$

Thus the dominant eigenvalue is $\lambda_2 = \frac{1+\sqrt{5}}{2}$ and its magnitude guarantees that the Fibonacci numbers $\{x_n\}$ form an increasing sequence. Since the eigenvalue λ_1 is negative, but of magnitude smaller than 1, its only effect is to superimpose a slight oscillation that dies out as n increases. It can be concluded that for large n the effect of λ_1 is negligible, so that

$$x_n \approx \frac{1}{\sqrt{5}}\lambda_2^n$$

So the ratio of the successive Fibonacci numbers x_{n+1}/x_n converges to

$$\frac{x_{n+1}}{x_n} = \lambda_2 = \frac{1+\sqrt{5}}{2}$$

This limiting value known as Golden mean is, therefore, given by

$$\tau = \frac{1+\sqrt{5}}{2} = 1.618033... \quad ... \quad (3.28)$$

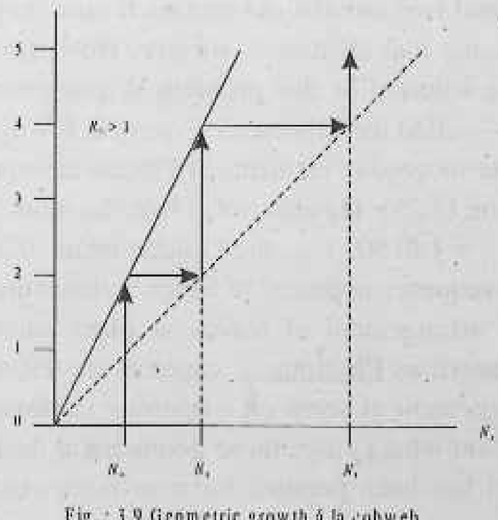


Fig. : 3.9 Geometric growth à la cobweb.

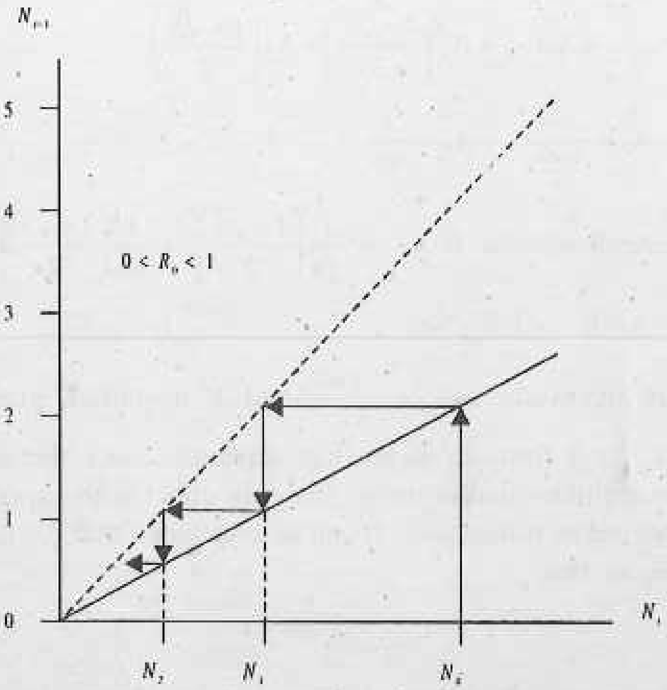


Fig. : 3.10 Geometric decay á la cobweb.

Historical Notes :

In 1202 Leonardo of Pisa (1175 – 1250), an Italian mathematician more affectionately known as Fibonacci (son of good nature), proposed the following problem (known as Rabbit Problem) : "Suppose that every pair of rabbits can produce only twice, when they are one and two months old and each time they produce exactly one new pair of rabbits. Assume that all rabbits survive. How many pairs will there be after n generation ?" The solution of this problem is a sequence of numbers (0, 1, 1, 2, 3, 5, 8, 13, 21 ....) — called the Fibonacci sequence. It was Kepler (1571 - 1630) who first observed that the successive elements of Fibonacci sequence satisfies the recursion or difference equation (3.25). He also noted that the ratio 2 : 1, 3 : 2, 5 : 3, 8 : 5........ approach the value τ = 1.618033 ..., the Golden mean. The manifestation of Golden mean and Fibonacci sequence appeared in Greek architecture, and in different biological forms. The regular arrangement of leaves or plant parts along the stem, apex or flower of a plant known as Phyllotaxis, captures the Fibonacci numbers. A striking example is the arrangement of seeds on a ripening sunflower. Biologists have not yet agreed conclusively on what causes these geometrical designs and patterns in plants, although the subject has been persued for over three centuries.

3.10 Summary

In this chapter we have discussed discrete-time models of populations on the basis of difference equations. The following are the problems discussed, (i) We have first discussed first-order linear difference equation with applications. (ii) We have next discussed the qualitative behaviors such as the stability of equilibrium (or fixed) points of first-order non-linear difference equations, (iii) We have then discussed the method of numerical and graphical solution namely Cobweb diagram method of finding the solution, determination of equilibrium points and their stabilities; (iv) We have next discussed second order linear difference equations and its application in the overlapping population growth, in particular ,in the study of Rabbit problem and Fibonacci sequence.

Exercise : A population obeys the following growth equation

$$x_{n+2} - 2x_{n+1} + 2x_n = 0$$

Find the population in generation n. Find the steady-state (if it exists). If so, is that stable?

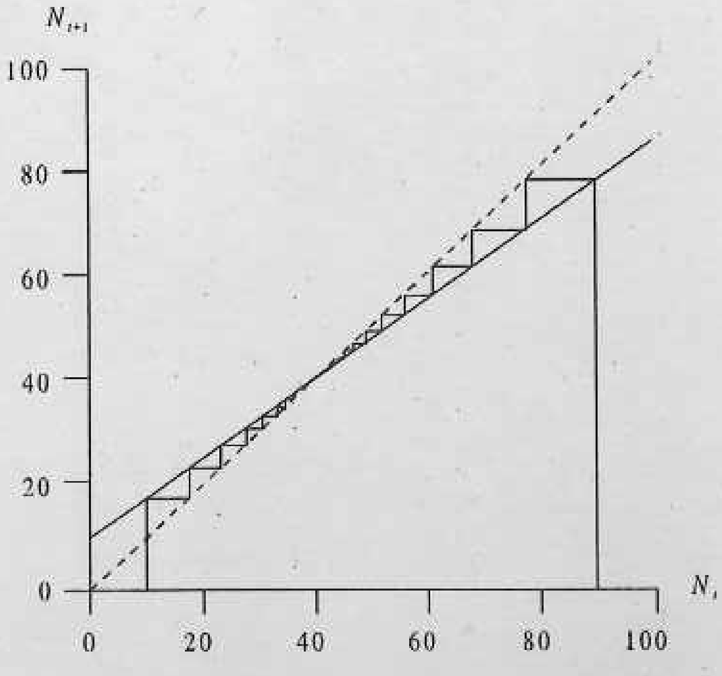


Fig. : 3.11 Cobwebbing to a stable equilibrium.

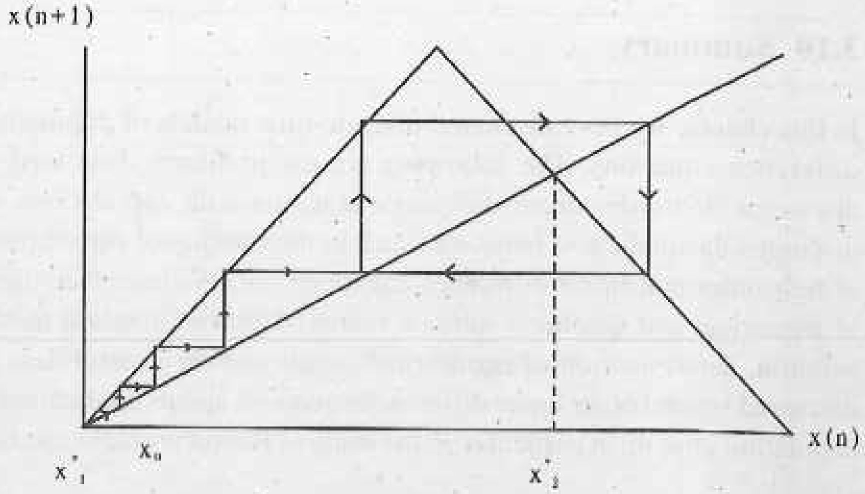


FIGURE : 3.12

Both equilibrium points $x^*_1 = 0$ and $x^*_2 = 2/3$ are unstable.

Unit 4 ❑ Delay Population Models

Objectives: The object of this chapter is to discuss delay differential equation model of population to take effect of time delay or time lag in the population growth.

Structure

4.1 ❑ Introduction

So far we have assumed that the rate at which a population is growing at time t depends on the magnitude of the population size (or density) at that same time. For example, consider the Malthus (or exponential) growth equation

$$\frac{dx(t)}{dt} = rx(t), \qquad x(o) = x_0 \qquad \ldots \qquad (4.1)$$

Now what happens if we know that the present growth rate depends, not on the present magnitude but on the magnitude at an earlier time? For example, the present growth rate of a colony of flies depends not on the number of flies right now but rather on the number of flies laying a certain number of eggs a week or so ago. In that case we write

$$\frac{dx(t)}{dt} = rx(t-\tau) \qquad \ldots \qquad (4.2)$$

where τ, the average incubation period of the eggs, is a time delay or time lag. As we shall see, this almost trivial change in the differential equation, greatly complicate the analysis and can produce drastic changes in the final answer.

4.2 ❑ Types of Delay Equations

There are two types of delay equations : (a) discrete-time delay equation and (b) distributed time delay equations.

(a) Discrete-time delay equation :
A more generalized delay equation for Malthus growth is

$$\frac{dx(t)}{dt} = r_1 x(t) + r_2 x(t-\tau) \quad \ldots \quad (4.3)$$

where r_1 and r_2 are constants. This is an example of differential - difference equations. Another important discrete-time delay equation is the logistic delay equation

$$\frac{dx(t)}{dt} = rx(t)\left(1 - \frac{x(t-\tau)}{x^*}\right)$$

where x^* is the carrying capacity.

(b) Distributed-time delay equation :

For distributed (or continuous) time delay, the logistic equation

$$\frac{dx(t)}{dt} = rx(t)\left(1 - \frac{x(t)}{x^*}\right)$$

becomes

$$\frac{dx(t)}{dt} = rx(t)\left(1 - \frac{1}{x^*}\int_{-\infty}^{t} K(t-\tau)x(\tau)d\tau\right) \quad \ldots \quad (4.6)$$

which is an example of integro-differential equation, the function K(t - x) is the delay function.

4.3 ❑ Discrete-time Delay Models

Consider a model equation of the form

$$\frac{dx(t)}{dt} = r(t)\, g(x(t-\tau)) \quad \ldots \quad (4.7)$$

An equilibrium point of the equation (4.7) is a value x^* such that $x^*\, g(x^*) = 0$ so that $x(t) = x^*$ is a constant solution of the differential-difference equation (4.7). The delay logistic equation (4.4) has two equilibrium points $x = 0$ and $x = x^*$.

Linearization about an Equilibrium Point:

Let us write $u(t) = x(t) - x^*$. Putting this in equation (4.7) and using Taylor series expansion, we have,

$$\frac{du(t)}{dt} = (x^* + u(t))\, g((x^* + u(t-\tau)$$

$$= (x^* + u(t)\, \{g(x^*) + g'(x^*)\, u(t-\tau) + \frac{g''(c)}{2!}u^2(t-\tau)\}$$

$= x^* g(x^*) + g(x^*) u(t) + x^* g'(x^*) u(t-\tau) + h(u(t), u(t-\tau))$

$= g(x^*) u(t) + x^* g'(x^*) u(t-\tau) + h(u(t), u(t-\tau))$

where c lies between x^* and $x^* + u(t-\tau)$ and $h(u(t), u(t-\tau) = g'(x^*) u(t) u(t-\tau) + x^* \frac{g''(c)}{2!} u^2(t-\tau)$ is a small quantiy when u(s) is small for $t-\tau \le s \le t$. So the differential-difference equation (4.7) reduces to the linear form

$$\frac{du(t)}{dt} = g(x^*) u(t) + x^* g'(x^*) u(t-\tau) \quad \ldots \quad (4.8)$$

obtained by neglecting the light order terms collected in $h(u(t), u(t-\tau))$. We have then the theorem :

Theorem 4.1: If all the solutions of the linear equation

$$\frac{du(t)}{dt} = g(x^*) u(t) + x^* g'(x^*) u(t-\tau) \quad \ldots \quad (4.9)$$

tends to zero as $t \to \infty$, then every solution x(t) of the equation (4.7) with $|x(t)-x^*|$ sufficiently small tends to the equilibrium point x^* as $t \to \infty$.

Asymptotic Stability :

For the differential equation

$$\frac{dx(t)}{dt} = xg(x) \quad \ldots \quad (4.9)$$

which is the case $\tau = 0$ of the equation (4.7) an equilibrium point x^* is asymptotically stable if and only if

$$\frac{d}{dx}(xg(x))\Big|_{x=x^*} = (xg(x))'\Big|_{x=x^*} = x^* g'(x^*) + g(x^*) < 0 \quad \ldots \quad (4.10)$$

So the equilibrium point $x^* = 0$ is asymptotically stable if $g(0) < 0$ and an equilibrium point $x^* > 0$ is asymptotically stable if $g'(x^*) < 0$, since $g(x^*) = 0$. The study of asymptotic stability of the fixed point x^* of the differential difference equation (4.7) is a bit difficult. It requires the condition

$$(xg(x))'\Big|_{x=x^*} < 0 \quad \ldots \quad (4.11)$$

This is, however, not sufficient for the delay equation (4.7). For this, another condition is required and this condition is provided by the theorem of linearization :

(a) For equilibrium point $x^* = 0$, the linearization is $u'(t) = g(o)\, u(t)$. Since $g(o) > 0$ for most models of this type, the equilibrium point $x^* = 0$ is unstable, since u(t) increases with time t.

(b) For equilibrium point $x^* > 0$, $g(x^*) = 0$ and the linearization is

$$\frac{dx(t)}{dt} = bu(t-\tau) \quad \ldots \quad (4.13)$$

where $b = x^* g'(x^*)$. In order to determine whether all solutions of the linear differential difference equation (4.13) tends to zero as $t \to \infty$ we take a solution of the form

$$u(t) = Ce^{\lambda t}, \text{ (C is a constant)} \quad \ldots \quad (4.14)$$

Putting this value in (4.13) we have,

$$\lambda = be^{-\lambda\tau} \quad \ldots \quad (4.15)$$

This is a transcendental equation for τ having infinitely many roots. A basic result, which we assume without proof, is that if all roots of the characteristic equation (4.15) have negative real parts, then all solutions of the differential - difference equation (4.13) tends to zero on $t \to \infty$. This result is analogous to the corresponding result for differential equation. However, it is very difficult to analyse transcendental equation (4.15) in the delay case. In the delay case, with $\tau > 0$, it is possible to show that the condition that all roots of the characteristic equation (4.15) have negative real parts is

$$0 < -b\tau < \frac{\pi}{2} \quad \ldots \quad (4.16)$$

The condition (4.16) implies $b < 0$ and in addition, that the time lag τ not to be too large. Combining the analysis with the above theorem we see that an equilibrium point $x^* > 0$ of the differential - difference equation

$$\frac{dx(t)}{dt} = x(t)\, g\,(x(t-\tau)) \quad \ldots \quad (4.17)$$

is asymptotically stable if

$$0 < -x^* g'(x^*)\tau < \pi/2 \quad \ldots \quad (4.18)$$

Example (4.1) : For delay—logistic equation

$$\frac{dx(t)}{dt} = rx(t)\left(1 - \frac{x(t-\tau)}{x^*}\right)$$

the stability condition (4.18) becomes,

$$0 < r\tau < \pi/2$$

Thus in addition to the stability condition. $(xg(x))'|_{x=x^*} < 0$ for ordinary differential equation, we must have additional requirement that the delay time τ be sufficiently small.

Example (4.2) : Show that the equilibrium point $x = K$ of the delay equation

$$\frac{dx(t)}{dt} = rx(t)\log\left(\frac{K}{x(t-\tau)}\right)$$

is asymptotically stable if $0 \leq r\tau < \pi/2$.

4.4 ❑ Distributed Delay Models :

In the previous section we have studied the discrete-time delay model of the type

$$\frac{dx(t)}{dt} = x(t)\,g\,(x(t-\tau))$$

This model equation can be generalized to the form

$$\frac{dx(t)}{dt} = x(t)\int_0^{\infty} g(x(t-s))\,p(s)\,ds \qquad \ldots \qquad (4.19)$$

describing a distributed delay. Here $p(s)\,ds$ represents the probability of a delay between s and $s + ds$, so that

$$\int_0^{\infty} p\,(s)\,ds = 1 \qquad \ldots \qquad (4.20)$$

The average delay is then, by definition

$$T = \int_0^{\infty} s\,p(s)\,ds \qquad \ldots \qquad (4.21)$$

<u>Definition</u>: An equilibrium point of the integro-differential equation

$$\frac{dx(t)}{dt} = x(t)\int_0^{\infty} g\,[x(t-s)]\,p(s)\,ds \qquad \ldots \qquad (4.22)$$

is a value x^* such that

$$x^*\int_0^{\infty} g(x^*)\,p(s)ds = x^*\,g(x^*) = 0$$

We see that $x^* = 0$ is an equilibrium point and equilibria $x^* > 0$ are given by

$$g(x^*) = 0 \qquad \ldots \qquad (4.23)$$

Linearization about an equilibrium point

To linearize (4.22) about an equilibrium point x^*, we put $u(t) = x(t) - x^*$ so that, we have,

$$\frac{du(t)}{dt} = (x^* + u(t))\int_0^\infty (g(x^*) + g'(x^*)\,u(t-s) + \ldots)\,p(s)ds$$

$$= (x^* + u(t))\left\{g(x^*) + g'(x^*)\int_0^\infty u(t-s)p(s)ds + \ldots\right\}$$

$$= x^* g(x^*) + g(x^*)\,u(t) + x^* g'(x^*)\int_0^\infty u(t-s)\,p(s)ds + \ldots \qquad \ldots \qquad (4.24)$$

As with other types of equations such as differential equations, difference equations, differential-difference equations, the behaviours of solution near an equilibrium point is thus described by the behaviours of solutions of the linearized equation. We are thus led to study the linear integro-differential equation of the form,

$$\frac{du(t)}{dt} = a\,u(t) + b\int_0^\infty u(t-s)\,p(s)ds \qquad \ldots \qquad (4.25)$$

where $p(s) \geq 0$, for $0 \leq s \leq \infty$ and $\int_0^\infty p(s)ds = 1$. To study the behaviour of solution of the equation (4.25) for a specific kernel p(s), we look for solution

$$u(t) = Ce^{\lambda t} \qquad \ldots \qquad (2.26)$$

and construct the characteristic equation,

$$\lambda = a + b\int_0^\infty e^{-\lambda s}\,p(s)ds = a + b\,L\{p(\lambda)\} \qquad \ldots \qquad (4.27)$$

where $L\{p(\lambda)\}$ is the Laplace Transform of the function p(s) evaluated at λ. We shall consider two specific choices of p, both normalized so that $\int_0^\infty p(s)=1$ and $\int_0^\infty sp(s)\,ds = T$ (average delay). We shall make use of the following formulae whenever necessary :

$$\text{(a)} \int_0^\infty e^{-\alpha s}ds = \frac{1}{\alpha}, \text{ (b)} \int_0^\infty se^{-\alpha s}ds = \frac{1}{\alpha^2}, \text{ (c)} \int_0^\infty s^2e^{-\alpha s}ds = \frac{2}{\alpha^3} \qquad \dots (4.28)$$

Example (4.3) : Let us take $p_1(s) = \frac{4s}{T^2}e^{-2s/T}$, so that p(o) = 0 and rising to a maximum at s = T/2, then falling exponentially. We have

$$L\{p_1(\lambda)\} = \int_0^\infty e^{-\lambda s}p_1(s)ds$$

$$= \frac{4}{T^2}\int_0^\infty se^{-(\lambda + 2/T)s}ds = \frac{4}{T^2\lambda^2 + 4T\lambda + 4}$$

The characteristic equation is

$$a + \frac{b}{T^2\lambda^2 + 4T\lambda + 4} = \lambda$$

$$\text{or, } \lambda^3 + \left(\frac{4T - aT^2}{T^2}\right)\lambda^2 + \left(\frac{4-4aT}{T^2}\right)\lambda - \frac{4(a+b)}{T^2} = 0 \qquad \dots \qquad (4.29)$$

The stability of the equilibrium requires all the roots of the polynomial equation (4.29) to have negative real parts. By Routh-Hurwitz condition all the roots of the cubic equation

$$\lambda^3 + \alpha\lambda^2 + \beta\lambda + \gamma = 0 \qquad \dots \qquad (4.30)$$

have negative real parts if and only if

$$\alpha > 0,\ \gamma > 0,\ \alpha\beta > \gamma \qquad \dots \qquad (4.31)$$

$$\text{Here } \alpha = \frac{4-aT}{T},\ \beta = \frac{4-4aT}{T^2},\ \gamma = -\frac{4(a+b)}{T^2}$$

and the stability conditions are

$$a+b<0,\quad aT<4 \text{ and } -bT<(2-aT)^2 \qquad \dots \qquad (4.32)$$

For the equation (4.25), $a = g(x^*)$, $b = x^* g'(x^*)$

(i) If the equilibrium is $x^* = 0$, then $b = x^* g'(x^*) = 0$. The condition of stability reduces to $a = g(x^*) < 0$ which is satisfied in population models only if there is an Allee effect (see Chapter - II).

(ii) If the equilibrium $x^* > 0$ then $a = 0$, since the equilibrium point satisfies the condition $x^* g(x^*)=0$. Then the stability condition (4.32) redues to $0 < -x^* g'(x^*) T < 4$.

Example (4.4) :

Take $P_2(s) = \frac{1}{T} e^{-s/T}$

Then $L\{p(\lambda)\} = \int_0^\infty e^{-\lambda s} p_2(s) ds$

$$= \frac{1}{T}\int_0^\infty e^{-\left(\frac{1}{T}+\lambda\right)s} ds = \frac{1}{\lambda T + 1}$$

The characteristic equation is

$$a + \frac{b}{\lambda T + 1} = \lambda$$

or, $\lambda^2 + \frac{1-aT}{T^2}\lambda - \frac{(a+b)}{T^2} = 0$

The stability condition that both the roots of this quadratic equation have negative real parts is

$1 - aT > 0$

$-(a+b) > 0$

From (4.24) we have $a = g(x^*)$, $b = x^* g'(x^*)$.

(i) If $x^* = 0$, then the stability condition reduces to $g(x^*) < 0$.which is not satisfied.

(ii) If $x^* > 0$, the stability condition is just $g'(x^*) < 0$, since $a = 0$. We thus see that for stability there is no requirement that the average delay T not be too large. Hence, in both the cases the stability criteria are satisfied independently of T. From the above example we come to the following conclusion :

"With distributed delay each delay kernel p must be examined in its own right. It is not true that increasing the average delav always destroys stability."

4.5. ❑ Summary:

In this chapter we have described both the types of discrete-time and continuous time delay models of populations. We have discussed the process of linearization about equilibrium and studied the criteria of stability for both the types of delay equations with illustrative examples.

Unit 5 ❑ Two-Species Models and Qualitative Analysis

Objectives: The object of this chapter is to make a qualitative analysis of two-species interacting model equations.

Structure

5.1 Introduction
5.2 Two-species Model Equation : Linearization and Stability
5.3 Periodic Solutions and Limit Cycles
5.4 Summary

5.1 ❑ Introduction

The model equations of interacting populations are usually non-linear. Analytical solutions of these equations are, in general, very difficult. The dynamical behaviors of such can be studied qualitatively. We can find out the criteria of stability of stationary (or equilibrium) states, we can find out the criteria of existence of periodic solutions and limit cycles without solving the equations exactly.

5.2 ❑ Two-Species Model Equations Linearization and Stability

We consider a population of two interacting species with population sizes (or densities) x(t) and y(t). As in the case of continuous single-species models, we assume that both x(t) and y(t) are continuously differentiable functions of time t. Let the model equations for the interacting system be of the form,

$$\frac{dx}{dt} = F(x, y) \quad \ldots \quad (5.1a)$$

$$\frac{dx}{dt} = G(x, y) \quad \ldots \quad (5.1b)$$

Although in models we neglect many factors of importance of real populations, they are useful first step and may represent real populations quite well.

Definition :

An equilibrium point (x^*, y^*) of the system of equations (5.1a) and (5.1b) is a solution of the equations $F(x, y) = 0$, $G(x, y) = 0$. Thus, an equilibrium is a constant solution of the system of equations (5.la) and (5.1b) Geometrically, an equilibrium is a point in the phase-space that is the orbit of a constant solution.

Linearization :

One of the main tools in studying continuous models for two interacting populations is linearization at equilibria, just as for models for single population. However, as linearization results can only give information about the behaviour of solutions near an equilibrium, they will not enable us to examine such questions as the existence of periodic orbits. However, for the study of local behaviours about the equilibrium the linearization is an important tool in dynamical theory.

Let us linearise the system of equations (5.1) about the equilibrium point (x^*, y^*). We write $u = x - x^*$ and $v = y - y^*$ and transform the system of equations (5.1) to the form

$$\frac{du}{dt} = G(x^* + u, y^* + v)$$

$$\frac{dv}{dt} = G(x^* + u, y^* + v)$$

Using Taylor's series expansion, we have

$$F(x^* + u, y^* + v) = F(x^*, y^*) + F_x(x^*, y^*)\,u + F_y(x^*, y^*)\,v + h_1$$

$$G(x^* + u, y^* + v) = G(x^*, y^*) + G_x(x^*, y^*)\,u + G_y(x^*, y^*)\,v + h_2$$

where h_1 and h_2 are functions that are small for small deviations (or perturbations) and v in the sense that

$$\lim_{\substack{u\to 0\\ v\to 0}} \frac{h_1(u,v)}{\sqrt{u^2+v^2}} = \lim_{\substack{u\to 0\\ v\to 0}} \frac{h_2(u,v)}{\sqrt{u^2+v^2}} = 0$$

Neglecting higher-order terms $h_1(u, v)$ and $h_2(u, v)$, we have the linear system of equations,

$$\frac{du}{dt} = F_x(x^*, y^*)\,u + F_y(x^*, y^*)\,v \quad \ldots \quad (5.2a)$$

$$\frac{du}{dt} = G_x(x^*, y^*)\,u + G_y(x^*, y^*)\,v \quad \ldots \quad (5.2b)$$

The coefficient matrix of the system (5.2)

$$A = \begin{bmatrix} F_x(x^*,y^*) & F_y(x^*,y^*) \\ G_x(x^*,y^*) & G_y(x^*,y^*) \end{bmatrix} \quad \cdots \quad (5.3)$$

is called the community matrix of the system at equilibrium (x^*, y^*). It describes the effect of the size of each species on the growth rates of itself and the other species at equilibrium. In matrix form the system of linear equation (5.2) can be written as

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \quad \cdots \quad (5.4)$$

$$\text{where} \quad \begin{matrix} a_{11} = F_x(x^*,y^*), & a_{12} = F_y(x^*,y^*) \\ a_{21} = G_x(x^*,y^*), & a_{22} = G_y(x^*,y^*) \end{matrix} \quad \cdots \quad (5.5)$$

are the elements of the community matrix A.

Stability of Equilibrium

Definition :

An equilibrium point (x^*, y^*) is said to be stable if every solution (x(t), y(t)) with (x(o), y(o)) sufficiently close to the equilibrium remains closed to the equilibrium for all $t > 0$. An equilibrium (x^*, y^*) is asymptotically stable if it is stable and if, in addition, the solution (x(t), y(t)) tends to the equilibrium (x^*, y^*) as $t \to \infty$. These definitions are natural extensions of the definitions given earlier for a single-species population.

Let us now find the explicit form of the criteria of stability. For this, we look for the solution of the linearized equations (5.4). The characteristic (or eigenvalue) equations of the system of linear equations (5.4) is given by

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0$$

or $\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}\,a_{22} - a_{12}\,a_{21}) = 0$

or $\lambda^2 - p\lambda + q = 0$ ... (5.6)

where $p = a_{11} + a_{22} = \text{Tr } A,\ q = (a_{11}\,a_{12} - a_{12}\,a_{21}) = \text{Det } A$ ... (5.7)

The stability of the equilibrium point (x^*, y^*) can be determined from the eigenvalues

λ of the system of equation (5.4) or the community matrix A given by (5.3). Infact, we have the theorem.

Theorem 5.1 :

The equilibrium (x^*, y^*) is asymptotically stable if the roots of the characteristic equation (5.6), that is, the eigenvalues λ have negative real parts. According to Routh-Hurwitz criterion the necessary sufficient conditions for the eigenvalues λ to have negative real parts are

$$p = \text{Tr } A < 0, \qquad q = \det A > 0 \qquad \ldots \qquad (5.8)$$

The conditions (5.8) are the sufficient conditions of asymptotic stability. The trace and determinant determine the eigenvalues λ. We classify the equilibrium points corresponding to the different nature of eigenvalues :

(i) If the two eigenvalues (λ_1, λ_2) are real and negative, the equilibrium is a stable node,

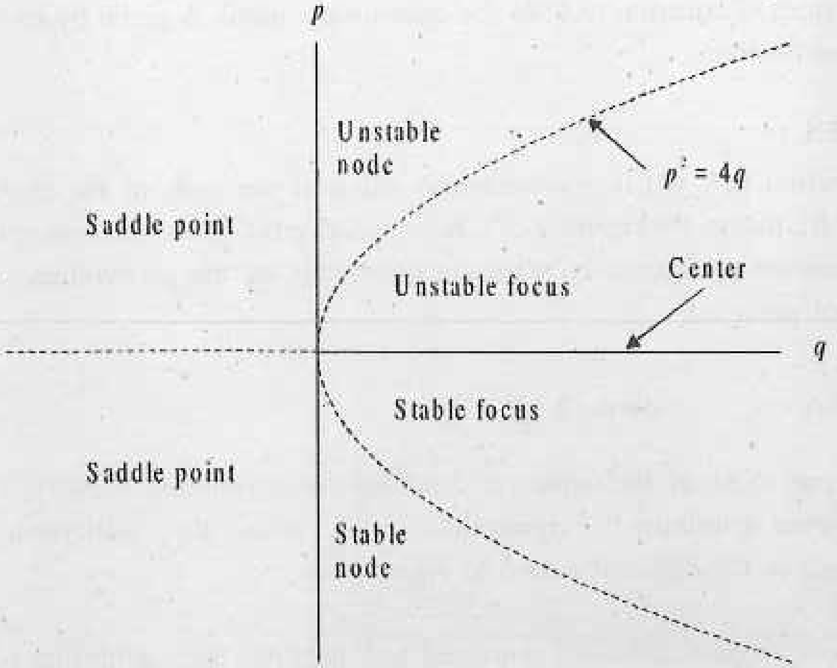
(ii) If the eigenvalues (λ_1, λ_2) are real and positive the equilibrium is an unstable node.

(iii) If (λ_1, λ_2) are real and of opposite sign, the equilibrium is a saddle point.

(iv) If the eigenvalues are complex with negative real parts, we have a stable focus,

(v) If the eigenvalues are complex with positive real parts, we have an unstable focus,

(vi) Finally, if the eigenvalues are purely imaginary, the linearised system will have a center but the original non-linear system will have a center or a stable or unstable focus, depending on the exact nature of the non-linear terms, [see Fig. (5.1)]



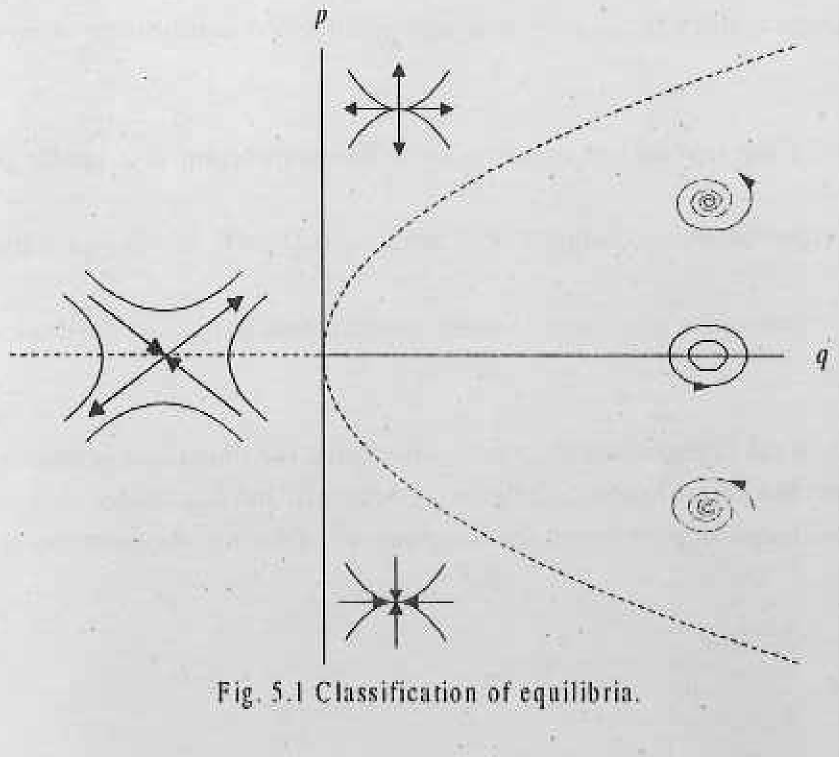


Fig. 5.1 Classification of equilibria.

Exercises :

1. The following two-dimensional non-linear differential equations have been proposed as a model of cell differentiation

$$\frac{dx}{dt} = y - x$$

$$\frac{dx}{dt} = \frac{5x^2}{4+x^2} - y.$$

(i) Determine the equilibrium points.

(ii) Linearize the system of equations each equilibrium point.

(iii) Determine the local stability of each positive equilibrium point and classify the equilibrium points.

2. The following system of equations (in dimensionless form) appear as a model of plant - harvivore system

$$\frac{dx}{dt} = 1 - kxy(y-1)$$

$$\frac{dx}{dt} = \varepsilon\, y\left(1 - \frac{y}{x}\right).$$

show that there is only one equilibrium and determine its stability.

5.3 ❑ Periodic Solution and Limit Cycles

In the preceding section we have analysed the behaviour of solutions starting near an equilibrium point. We now consider the case where the solution does not begin near the equilibrium; in particular we want to examine the behaviour of solutions of systems that have no asymptotic stable equilibrium. Such system can arise in the models of predator-prey system. We consider the two-dimensional system

$$\frac{dx}{dt} = F(x,y)$$

$$\frac{dx}{dt} = G(x,y).$$

Definition :

Let (x(t), y(t)) be a solution that is bounded as $t \to \infty$. The positive semi-orbit C^+ of this solution is defined to be the set of points (x(t), y(t)) for $t \geq 0$ in the (x, y) plane. The limit set $L(C^+)$ of the semi-orbit is defined to be the set of all points $(\bar{x}, \bar{y})$ such that there is a sequence of times $t_n \to \infty$ with $x(t_n) \to \bar{x}, y(t_n) \to \bar{y}$ as $n \to \infty$. For example, if the solution $(x(t), y(t))$ tends to an equilibrium point (x^*, y^*) as $t \to \infty$, then the limit set consists of the equilibrium point (x^*, y^*). If $(x(t), y(t))$ is a periodic solution so that the semi-orbit C^+ is a closed curve, then the limit set $L(C^+)$ consists of all points of the semi-orbit C^+. It is not difficult to show that the limit set of a bounded semi-orbit is closed, bounded and a connected set.

Definition :

An invariant set for the system (5.9) is a set of points in the plane which contains the positive-semi orbit through every point of the set. Thus, for example, an equilibrium is an invariant set, and a closed orbit corresponding to a periodic solution is an invariant set. It is possible to prove, making use of continuous dependence of solutions of differential equation on initial conditions, that the limit set of a bounded semi-orbit is an invariant set.

The results stated above are valid for autonomous differential equations in all dimensions, but in two dimensions more information on the structure of limit sets is available. The reason for this involves the topological properties of the plane, especially the Jordan curve theorem which states that a simple closed curve in the plane divides the plane into two disjoint regions - which is not valid in more than two dimensions. The fundamental result on the behaviour in the large of solutions of autonomous systems in the plane is the Poincare′ — Bendixson theorem.

Theorem 5.2 (Poincare′ - Bendixson Theorem :

If C^+ is a bounded semi-orbit whose limit set $L(C^+)$ contains no equilibrium points, then either C^+ is a periodic orbit and $L(C^+) = C^+$ or $L(C^+)$ is a periodic orbit, called a limit cycle, (which C^+ approaches spirally, either from inside or from outside.) We shall not go to the proof of the theorem, which may be found in many standard books on differential equations and dynamical system. We conclude the section by stating a theorem - due to Bendixson giving a criterion implying that there can not be a periodic orbit in a given region.

Theorem 5.3 : (Bendixson's negative criterion) :

Consider the system (5.9), that is

$$\frac{dx}{dt} = F(x, y), \quad \frac{dy}{dt} = G(x, y)$$

where F and G are continuously differentiable function of (x, y) defined on some simply connected domain $D \subset R^2$ (by simply connected, we mean that the domain has no 'holes' or disjoint portion.) If

$$\nabla.(F,G) = \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}\right) \quad \ldots \quad (5.10)$$

is of one sign in D, there can not be a closed orbit contained in D.

Proof : The proof is by contradiction . Suppose that we do have a closed orbit C, with interior $\mathbb{I}$, contained in D that satisfies the equations (5.9). Suppose also that the right-hand side of equation (5.10) is of one sign. If follows that

$$\iint_{\mathbb{I}} \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}\right) dxdy \neq 0 \quad \ldots \quad (5.11)$$

Applying Green's theorem we transform the surface integrat (5.11) to the form of line integral. we have then

$$\oint_c (F\,dy - G\,dx) \neq 0 \quad \ldots \quad (5.12)$$

The last integral may be written as

$$\oint_c \left(F\frac{dy}{dt} - G\frac{dx}{dt}\right) dt \quad \ldots \quad (5.13)$$

and since C has been assumed to satisfy system (5.9), we may transform (5.13) to the form

$$\oint_c (FG - GF)dt = \oint_c 0.dt = 0 \quad \ldots \quad (5.14)$$

which contradict (5.12); and so we have been mistaken in assuming the existence of a closed orbit C (contained in D) that satisfies the system of equations (5.9). If the divergence is of one sign, there can not be such an orbit.

Remark : Since a periodic solution corresponds to a closed orbit and vice-versa, the above theorem also provides the criterion of existence of a periodic solution of the system of equations (5.9).

The French mathematician H. Dulac made the useful observation that the last system (5.9) is a member of the family of dynamical systems

$$\left.\begin{aligned}\frac{dx}{dt} &= B(x,y)\,F(x,y)\\ \frac{dy}{dt} &= B(x,y)\,G(x,y)\end{aligned}\right\} \quad \ldots \quad (5.15)$$

that share the same phase - portrait. If one can disprove the existence of a closed orbit for any member of the family, one can disprove the existence of a closed orbit for every member of the family (5.15). This leads to a minor, but powerful extension of Bendixson's negative criterion.

Theorem 5.4 : (Bendixson-Dulae negative criterion) :

Let B be a smooth function on $D \subset R^2$ (with all other assumptions as before). If

$$\nabla.(F,G) = \frac{\partial(BF)}{\partial x} + \frac{\partial(BG)}{\partial y} \quad \ldots \quad (5.16)$$

is of one sign, then no closed orbit contained within D.

The above theorem does not tell us how to find B(x, y). There is no general method for constructing B. However, we are lucky enough to find such a function.

Example (5.1) : Consider the system

$$\frac{dx}{dt} = x(1-x-y)$$

$$\frac{dx}{dt} = \beta(x-\alpha)y.$$

Let $B = \frac{1}{xy}$, then $BF = \frac{1-x-y}{y}$, $BG = \frac{\beta(x-\alpha)}{x}$ and $\frac{\partial(BF)}{\partial x} + \frac{\partial(BG)}{\partial y} = -\frac{1}{y}$

The last expression is strictly negative in the interior of the first quadrant of the (x, y)-plane. Thus, there cannot be a closed orbit that satisfies the above system of equations and that lies entirely within the interior of the first-quadrant. So, the system of equation contains no periodic solution within the first quadrant.

Example (5.2) : Investigate the qualitative behaviour of the solution of the system

$$\frac{dx}{dt} = x\left(1-\frac{x}{30}\right) - \frac{xy}{x+10}$$

$$\frac{dx}{dt} = y\left(\frac{x}{x+10} - \frac{1}{y}\right)$$

Solution : There are three equilibrium points $(x^*, y^*) = (0, 0), (30, 0), (5, 12.5)$. The community matrix at (0, 0) is

$$A=\begin{pmatrix} 1 & 0 \\ 0 & -1/3 \end{pmatrix}$$

Since det A < 0, the equilibrium point (0, 0) is unstable. The community matrix at (30, 0) is

$$A=\begin{pmatrix} -1 & -3/4 \\ 0 & -5/12 \end{pmatrix}$$

This equilibrium (30, 0) is also unstable, since det A < 0.
The community matrix at (5,12.5) is

$$A=\begin{pmatrix} 1/6 & -3/4 \\ 5/9 & 0 \end{pmatrix}$$

Here tr A> 0, det A > 0 so (5, 12.5) is also unstable. If we add the two equations of the model, we obtain

$$\frac{d}{dt}(x+y)=x\left(1-\frac{x}{30}\right)-\frac{1}{3}$$

Thus (x + y) is decreasing except in the bounded region $\frac{y}{3}<\left(1-\frac{x}{30}\right)$. In order that an orbit be unbounded, we must have (x + y) unbounded. However, this is impossible, since (x + y) is decreasing whenever (x + y) is large. Thus every orbit of the system is bounded. Since all equilibria are unstable, the Poineare'-Bendixson theorem implies that there must be a periodic orbit arround (5, 12.5) to which every orbit tends as $t \to \infty$.

Exercises :

(1) Determine the behaviour of the solutions in the first quadrant of the system

$$\frac{dx}{dt}=x(100-4x-2y)$$

$$\frac{dy}{dt}=y(60-x-y)$$

(2) Consider the system

$$\frac{dx}{dt}=x(ax+by)$$

$$\frac{dy}{dt} = y(cx + dy)$$

(i) show that every trajectory with $x(0) \geq 0$, $y(0) \geq 0$ satisfies $x(t) \geq 0$, $y(t) \geq 0$ for all $t > 0$ (i.e. trajectories starting in the first quadrant remain in the first quadrant foroever

(ii) use Dulac criterion with $B(x, y) = 1/xy$ to show that there are no periodic orbits if $ac > 0$.

***Mathematical Note : Routh-Hurwitz Criterion**

It is difficult or impossible to find explicitly all the roots of the characteristic equation of a multi-dimensional system. There is, however, a general criterion for determining whether all roots of a polynomial equation have negative real parts. This criterion known as Routh-Hurwitz criterion gives conditions on the coefficient of a polynomial equation

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_{n-1}\lambda + a_n = 0$$

under which all roots have negative real parts. For $n = 2$, the Routh-Hurwitz condition are

$$a_1 > 0, \quad a_2 > 0.$$

which is equivalent to the conditions : $\text{tr } A < 0$, $\det A > 0$. For $n = 3$, the conditions are

$$a_3 > 0, \quad a_1 > 0, \quad a_1 a_2 > a_3$$

For a polynomial of degree n, there are n conditions. This criterion is useful on occasions, it is, however, complicated for problem of many dimensions.

5.4 ❑ Summary:

This chapter consists of two parts:

(i) In the first part we have introduced general model equations of two interacting species, we have found out equilibrium states and reduced model equations to the linear form. On the basis of linear equations we have investigated the criteria of asymptotic stability (local) of the system.

(ii) The second part deals with the systems that have no asymptotic stable equilibria. For such systems we have investigated the criteria of existence of periodic solutions and limit cycles.

Unit 6 ❑ Two-Species Models : Lotka-Volterra Systems

Objectives: The chapter consists of an account of Lotka-Volterra dynamical models of interacting poplations.

Structure

6.1 ❑ Introduction

When two or more species interact the population dynamics of each species is effected. In general, the is a whole web of interacting species, called the tropic web which make structurally complex communities. The dynamical models of such interacting species are provided by Lotka-Volterra systems of equations and there are three types of model equations dealing with interactions. Three models are (i) predator-prey model (ii) competition model and (iii) mutualistic model.

In this chapter we shall study the dynamical processes involved in each type of model systems.

6.2 ❑ Predator-Prey Models

6.2.1 Classical Lotka-Volterra Model

Let us consider a prey-predator system. Let x(t) be the number (or density) of prey and y(t) be the number (or density) of predators. Lotka-Volterra model equations for the system are

$$\frac{dx}{dt} = \lambda x - bxy \quad \ldots \quad (6.1a)$$

$$\frac{dy}{dt} = -\mu y + cxy \quad \ldots \quad (6.1b)$$

The first term on the right-hand side of (6.1a) implies that the prey will grow exponentially in the absence of the predator : the prey are limited by predator. The second term describes the loss of prey due to predators. This loss is assumed to be proportional to both the numbers of prey and predators, resulting in what is often described as a mass-action term. Turning to the right hand side of equation (6.1b), we see that the loss of prey heads to the production of new predators, and that the predator population decreases exponentially in the absence of prey. The system of equations (6.1) cannot be solved analytically, but we can obtain some information about the behaviour of its solutions by studying the orbits or trajectories of solutions in the (x, y) plane. Eliminating t from the Lotka-Volterra equations (6.1), we have

$$\frac{dy/dt}{dx/dt} = \frac{dy}{dx} = \frac{y(-\mu + cx)}{x(\lambda - by)}$$

We may solve this equation by separation of variables

$$\int \left(\frac{-\mu + cx}{x}\right) dx = \int \left(\frac{\lambda - by}{y}\right) dy$$

or $-\mu \log x - \lambda \log y + ex + by = h$

where h is a constant of integration. Let us write

$$V(x, y) = -\mu \log x - \lambda \log y + cx + by \quad \ldots \quad (6.2)$$

So that the orbit of the system is given by

$$V(x,y) = h \quad \ldots \quad (6.3)$$

The minimum of the function V(x, y) is given by $\frac{\partial V}{\partial x} = 0$ and $\frac{\partial V}{\partial y} = 0$, that is, by

$(x, y) = \left(\frac{\mu}{c}, \frac{\lambda}{b}\right)$. This is also the equilibrium position (x*, y*) of the Lotka-Volterra

system (6.1), that is, $x^* = \frac{\mu}{c}, y^* = \frac{\lambda}{b}$.

So $V(x,y)\big|_{min} = V(x,y)\big|_{\substack{x=x^* \\ y=y^*}} = -\mu \log\frac{\mu}{c} - \lambda \log\frac{\lambda}{b} + \mu + \lambda = h_0$

Every orbit of the system is given implicitly by an equation V(x, y) = h for some constant $h \geq ho$, which is determined by some initial conditions. We make the change of variables :

$$x = x^* + u = \frac{\mu}{c} + u$$

$$y = y^* + v = \frac{\lambda}{c} + v$$

Then V(x, y) becomes,

$$V(x,y) = -\mu \log\left(\frac{\mu}{c} + u\right) - \lambda \log\left(\frac{\lambda}{b} + v\right)$$

$$+ c\left(\frac{\mu}{c} + u\right) + b\left(\frac{\lambda}{b} + v\right) = h$$

We observe that

$$\log\left(\frac{\mu}{c} + \mu\right) = \log\frac{\mu}{c} + \log\left(1 + \frac{cu}{\mu}\right)$$

If (h - ho) is small, we may use the approximation $\log(1 + x) \sim = x - \frac{x^2}{2}$ so that, we have

$$\log\left(\frac{\mu}{c} + u\right) = \log\frac{\mu}{c} + \frac{cu}{\mu} - \frac{c^2u^2}{\mu^2}$$

$$\log\left(\frac{\lambda}{b} + v\right) = \log\frac{\lambda}{b} + \frac{bu}{\lambda} - \frac{b^2c^2}{\lambda^2}$$

Then the orbits V(x, y) = h are approximated by

$$-\mu \log\frac{\mu}{c} - cu + \frac{c^2}{\mu}u^2 - \lambda \log\frac{\lambda}{b} - bv + \frac{b^2}{\lambda}v^2 + \mu + cu + \lambda + bv = h$$

or, $$\left(\frac{c^2}{\mu}\right)u^2 + \left(\frac{b^2}{\lambda}\right)v^2 = h + \mu \log\frac{\mu}{c} + \lambda \log\frac{\lambda}{b} - \mu - \lambda = h - h_0 \ldots \qquad (6.4)$$

which represents an ellipse (if $h > h_0$) with equilibrium $(x^*, y^*) = \left(\frac{\mu}{c}, \frac{\lambda}{b}\right)$ as its centre.

This shows that for $(h - h_0)$ small and positive the orbits are closed curves around the equilibrium point, since the solutions run around closed orbit they must be periodic. It is easy to see that the maximum prey population comes one quarter of a cycle before the maximum predator population (see Fig. 6.1).

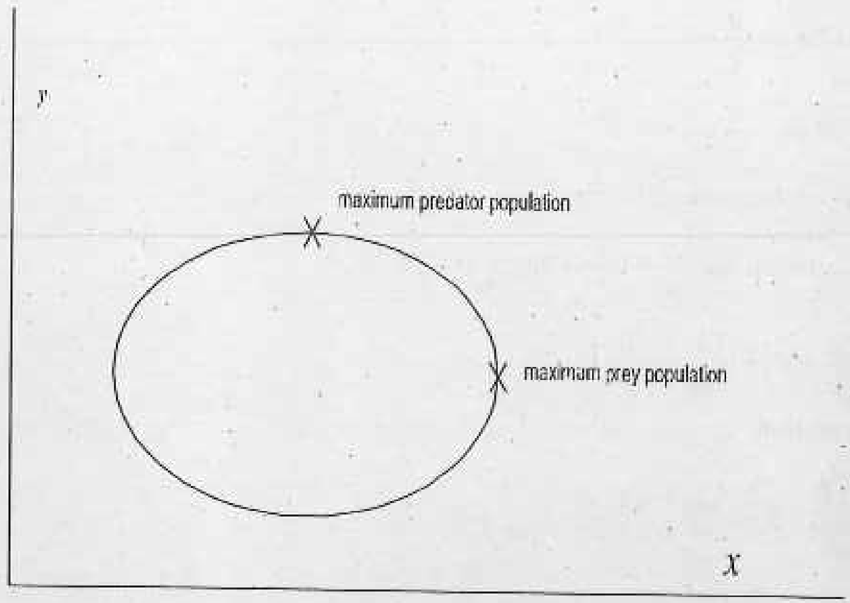


Figure : 6.1

Historical Background :

How did this model (Volterra model) arise ? In the mid-1920s Umberzto D'Ancona, an Italian marine biologist, performed a statistical analysis of the fish that were sold in the markets of Trieste, Fiume and Venice between 1910 and 1923. Fishing was largely suspended in the upper Adriatic during the First World War, from 1914 to 1978. D'Ancona observed the increase of relative frequency of some species like Selachians (old name of Sharks and Shark - like fish) during the war years and decrease with the increase of fishing. The relative abundance of prey, in turn, followed the opposite pattern. Why did this happen? At that time Umberto was engaged to Luisa Volterra an ecologist. Umberto posed this question to Vito Volterra, his future father-in-law and a famous mathematician. Volterra (1926) constructed a model known as Lotka-Volterra model (because A. J. Lotka (1925) constructed a similar model in a different context about the same time) based on the assumption of that fish and sharks were in predator-prey relationship.

Example (6.1): Show that the period of Oscillation of prey and predator population sizes of Lotka-Volterra system is $2\pi/\lambda\mu$

Solution : Linearing the system of equation (6.1) about the non-trivial equilibrium $(\mu/c, \lambda/b)$ we have the linear equations

$$\frac{du}{dt} = -\frac{b\mu}{c}V \qquad \ldots \qquad (6.5a)$$

$$\frac{dv}{dt} = -\frac{\lambda c}{b}u \qquad \ldots \qquad (6.5b)$$

It is easy to combine both the equations (6.5a) and (6.5b) to give

$$\frac{d^2u}{dt^2} = -\lambda\mu\ u \qquad \ldots \qquad (6.6)$$

which is the equation of a simple-harmonic Osillator with frequency $\frac{2\pi}{\lambda\mu}$.

6.2.2 Predator-Prey System : A Realistic Model

The Lotka-Volterra model represented one of the triumphs of early attempts at mathematical modelling in population biology. The dynamics of predator-prey system modelled by Volterra is interesting, but it is unrealistic and there are some flaws in the model. The model is structurally unstable and is extremely sensitive to perturbation. A small change in the initial population size may produce a change to a different periodic orbit, while the addition of a perturbing term to the system of differential equation may eliminate the balanced neutrally stable family of periodic orbits that we have observed. We, therefore, need to look at other predator-prey models.

We now consider a more realistic model of predator-prey system by assuming that in the absence of predators, the prey species grow logistically

$$\frac{dN}{dT} = rN\left(1 - \frac{N}{k}\right) - cNP \qquad \ldots \qquad (6.7)$$

$$\frac{dN}{dT} = bNP - mP$$

where N is the prey-population size, P that of predators, we have written time as T (rather than t) because we will soon rescale this variable. To ease the analysis, we non-dimensionalize all the variables step and by step.

We use the first dimensionless variable

$$x = \frac{N}{K} \qquad \ldots \qquad (6.8)$$

Then the system (6.7) becomes

$$\frac{dx}{dT} = rx(1-x) - cxP \qquad \ldots \qquad (6.9a)$$

$$\frac{dP}{dT} = bkxP - mP \qquad \ldots \qquad (6.9b)$$

To simplify (6.9b), we use the dimensionless variable

$$y = \frac{c}{r}P \qquad \ldots \qquad (6.10)$$

and eliminate P, to have

$$\frac{dx}{dT} = rx(1-x-y) \qquad \ldots \qquad (6.11a)$$

$$\frac{dy}{dT} = bkxy - my \qquad \ldots \qquad (6.11b)$$

Finally, we write $t = r\,T$ ... (6.12)

and note that $\frac{d}{dt} = \frac{d}{dt}.\frac{d}{dt} = r\frac{d}{dt}$ ... (6.13)

Then the system of equations (6.1 la) and (6.11b) takes the form

$$\frac{dx}{dt} = x(1-x-y) \qquad \ldots \qquad (6.14a)$$

$$\frac{dy}{dt} = \frac{bk}{r}y\left(x - \frac{m}{bk}\right) \qquad \ldots \qquad (6.14b)$$

we write $\alpha = \frac{m}{bk}$, $\beta = \frac{bk}{r}$ ... (6.15)

With these, we have the system of equations

$$\frac{dx}{dt} = x(1-x-y) \qquad \ldots \qquad (6.16a)$$

$$\frac{dx}{dt} = \beta(x-\alpha)y \qquad \ldots \qquad (6.16b)$$

The simplified model (6.16) has three equilibria : $(x^*, y^*) = (0, 0), (1, 0), (\alpha, 1-\alpha)$. These equilibria occur at the intersections of the prey and predator zero growth isoclines*. [see Fig. (6.2)]

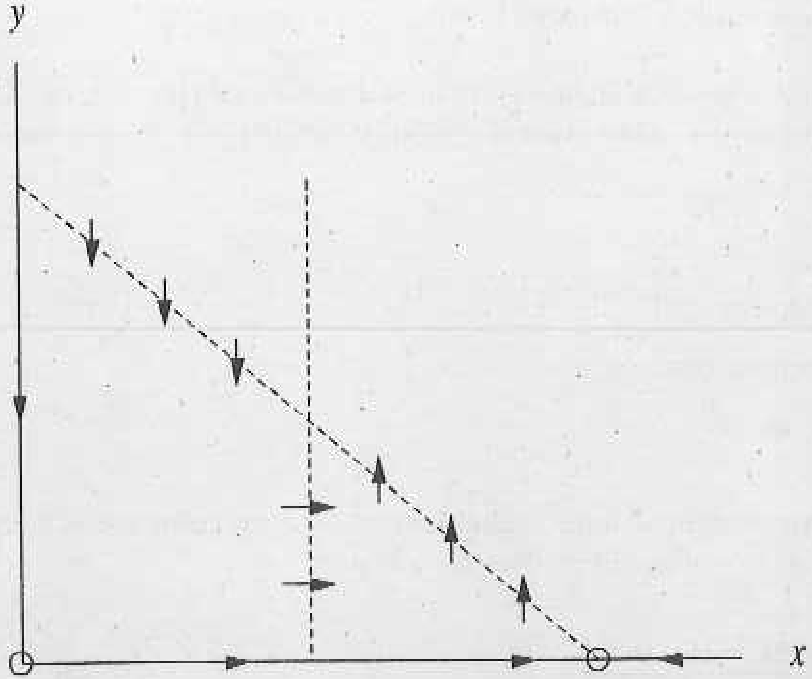


Fig. (6.2) : Predator and Prey zero growth isoclines.

[Zero-growth isoclines are the curves in the (x, y) plane or phase-plane along which $\frac{dx}{dt}=0$ and $\frac{dy}{dt}=0$. These curves should properly be called nullclines.]

The community matrix of the system (6.16) is

$$A=\begin{bmatrix}1-2x-y & -x\\ \beta & \beta\,(x-\alpha)\end{bmatrix}_{(x^*,y^*)}$$

At $(x^*, y^*) = (0, 0)$ the community matrix is

$$A=\begin{bmatrix}1 & 0\\ 0 & -\beta\alpha\end{bmatrix} \qquad \ldots \quad (6.18)$$

Eigenvalues $\lambda = 1, -\beta\alpha$

Let det $A = -\beta\alpha < 0$. So (0, 0) is a saddle point (Fig. 6.3). At $(x^*, y^*) = (1, 0)$, the community matrix is

$$A=\begin{bmatrix}-1 & -1\\ 0 & \beta(1-\alpha)\end{bmatrix} \qquad \ldots \quad (6.19)$$

Eigenvalues $= -1, \beta(1-\alpha)$, $\det A = \beta(\alpha - 1)$.

If $\alpha > 1$, $\det A > 0$ which implies $(1, 0)$ to be a stable node. If $\alpha < 1$, $\det A < 0$ which implies $(1, 0)$ to be a saddle point (Fig. 6.4) At $(x^*, y^*) = (\alpha, 1-\alpha)$, the community matrix is:

$$A = \begin{bmatrix} -\alpha & -\alpha \\ \beta(1-\alpha) & 0 \end{bmatrix} \qquad \ldots \quad (6.20)$$

The characteristic equation is

$$\lambda^2 + \alpha\lambda + \alpha\beta(1-\alpha) = 0 \qquad \ldots \quad (6.21)$$

By Routh-Hurwitz criterion the equilibrium $(\alpha, 1-\alpha)$ is stable if $\alpha < 1$ end unstable if $\alpha > 1$ (Fig. 6.5). The eigenvalues are given by

$$\lambda = \frac{-\alpha \pm \{\alpha^2 - 4\alpha\beta(1-\alpha)\}^{1/2}}{2} \qquad \ldots \quad (6.22)$$

If we examine the discriminant, we see that we have a node if

$$\alpha > \frac{4\beta}{1+4\beta} \qquad \ldots \quad (6.23)$$

and a focus if

$$\alpha < \frac{4\beta}{1+4\beta} \qquad \ldots \quad (6.23)$$

The model does not show any periodic orbits — in contrast to classical Lotka-Volterra system. The addition of a small amount of prey-density dependence has destroyed the family of periodic orbits that we have observed in the classical Lotka-Volterra model. It leads to the conclusion that the classical Lotka-Volterra system is structurally unstable.

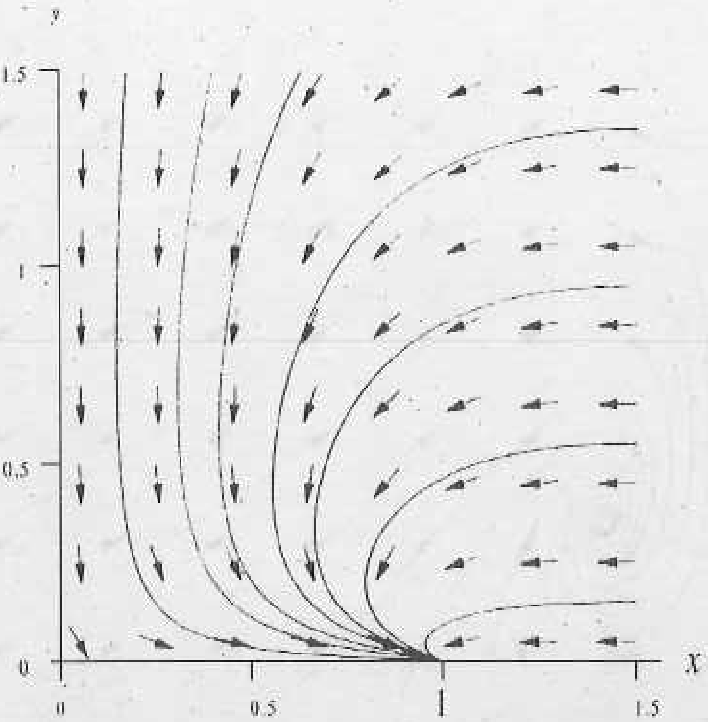


Fig. 6.3 Phase portrait for $a > 1$

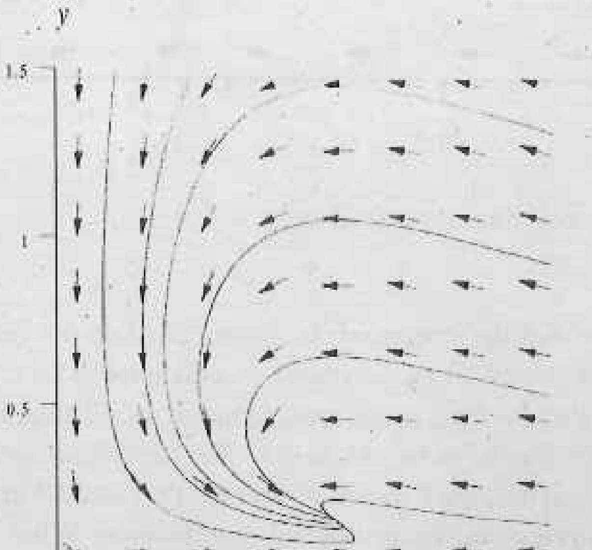


Fig. 6.4 Phase portrait for $a < 1$, $a > 4\beta/(1+4)$

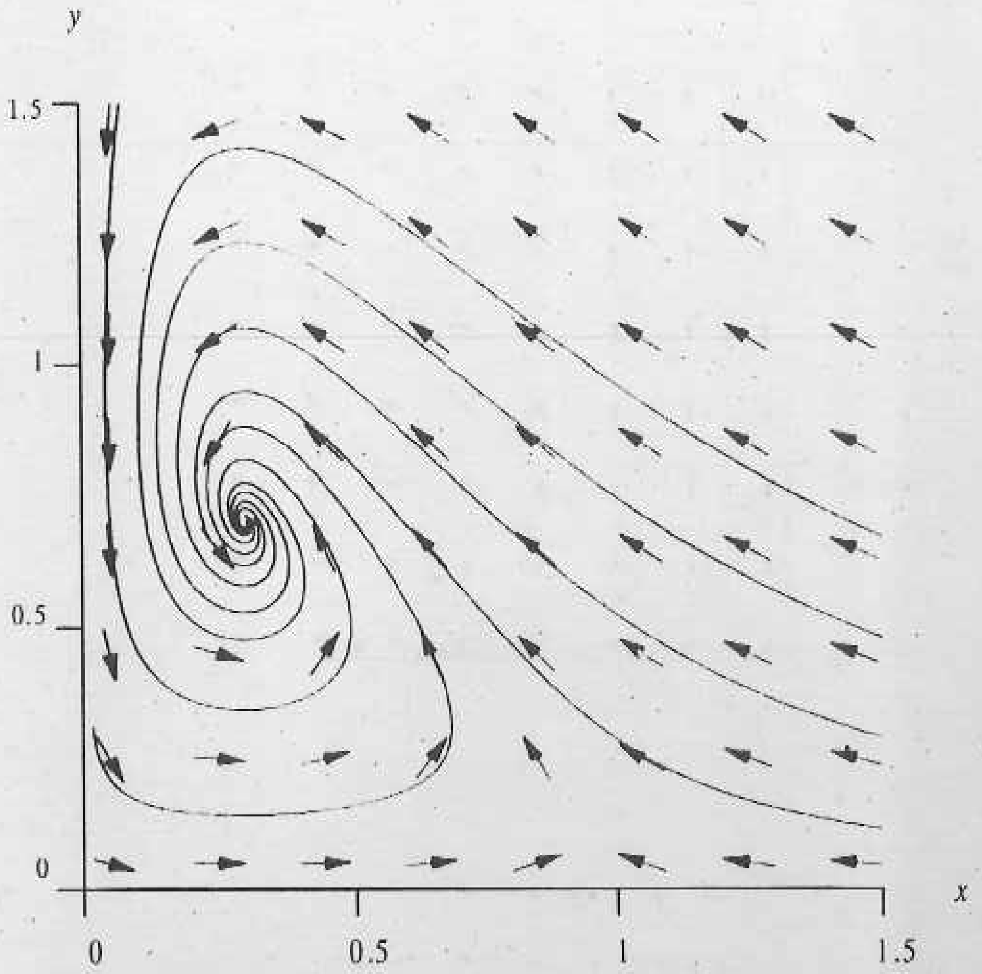


Fig. 6.5 Phase portrait for $a < 1$, $a < 4\beta(1+4)$

Example (6.2) : Show that the system of equations (6.16) do not contain any periodic solution in the first quadrant of (x, y) plane, (see example (5.1), Chapter-V).

Remarks : Predator-Prey system is the most dynamic of all interacting populations. There are many other things to be discussed. We have discussed the criterion of periodic orbit and its significance in the predator - prey model equations. We have not discussed limit cycles and its ecological significance. What biological factors create limit cycles? The inclusion of a more realistic functional response is one such factor. The functional response is the rate at which predator captures prey. Heretofore, the functional response was a linearly increasing function of prey density. However, predator may become satiated. They may also be limited by the handling time of catching and consuming their prey. This limit on the predator's ability can have a profound effect on the dynamics of a predator-prey model. There are four different functional response curves. Predator - Prey models with functional responses can exhibit limit cycles, bifurcation and chaotic behaviour in the dynamics of the systems. These are, however beyond the introductory course.

Example (6.3) :

Determine the qualitative behaviour of a predator-prey system modelled by

$$\frac{dx}{dt} = x\left(1 - \frac{x}{30}\right) - \frac{xy}{x+10}$$

$$\frac{dy}{dt} = y\left(\frac{xy}{x+10} - \frac{1}{3}\right)$$

Solution :

We have studied this system in Example (5.2) of Chapter - V and shown that every orbit approaches a periodic orbit around the (unstable) equilibrium (5, 12.5). Thus the species co-exist with oscillations.

Exercises :

(1) Determine the equilibrium behaviour of a predator-prey system modelled by

$$\frac{dy}{dt} = y\left(\frac{x}{x+10} - \frac{3}{5}\right)$$

$$\frac{dx}{dt} = x\left(1 - \frac{x}{13}\right) - \frac{xy}{x+10}$$

(2) Show that the equilibrium (x^*, y^*) with $x^* > 0$, $y^* > 0$ of the predator-prey model

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \frac{axy}{x+A}$$

$$\frac{dy}{dt} = sy\left(\frac{ax}{x+A} - \frac{aB}{A+B}\right)$$

is unstable if $k > A + 2B$ and asymptotically stable if $B < k < A + 2B$.

(3) Investigate the stability of the equilibrium of the chemostat modelled by the equations

$$\frac{dy}{dt} = \frac{acy}{C+A} - ay$$

$$\frac{dc}{dt} = q(c^{(b)} - c) - \frac{\beta acy}{C+A}$$

where y is the number of bacteria and c is the concentration of nutrient in the chemostat.

6.3 ❑ Competition Models

6.3.1 Lotka-Volterra Classical Competition Model

We consider the classical model of competition due to Lotka and Volterra. The Lotka-Volterra competition model is an interference competition model : two species are assumed to diminish each others per capita growth rate by competition.

We begin with two species, with population sizes $x_1(t)$ and $x_2(t)$ at any time t. We assume that each species grows logistically in the absence of the other. The model equatiors are

$$\frac{1}{x_1}\frac{dx_1}{dt} = r_1\left(1 - \frac{x_1}{k_1} - \frac{\alpha_{12}}{k_1}x_2\right) \qquad \dots \quad (6.25a)$$

$$\frac{1}{x_2}\frac{dx_2}{dt} = r_2\left(1 - \frac{x_2}{k_2} - \frac{\alpha_{21}}{k_2}x_1\right) \qquad \dots \quad (6.25b)$$

Each individual of the second species causes a decrease in the per capita growth of the first species; and vice versa. To parameterize this effect, we have introduced a pair of competition coefficients α_{12} and α_{21}, that describe the strength of the effect of the species 2 on the species 1 and of species 1 on the species 2 respectively. The system of equations (6.25) can be rewritten as

$$\frac{dx_1}{dt} = \frac{r_1}{k_1}x_1(k_1 - x_1 - \alpha_{12}x_2) \qquad \dots \quad (6.26a)$$

$$\frac{dx_2}{dt} = \frac{r_1}{k_2}x_2(k_2 - x_2 - \alpha_{21}x_1) \qquad \dots \quad (6.26b)$$

The complete characterization of the dynamics of the equations (6.26) revolves around the orientations of zero-growth isoclines. The x_2 − zero growth isoclines given by $\frac{dx_2}{dt} = 0$, are

$$x_2 = 0 \qquad \dots \quad (6.27a)$$

and $x_2 = k_2 - \alpha_{21}x_1$ ... (6.27b)

[see Fig. (6.6)]. Below the line given by (6.27b), x_2 increases; above this line, x_2 decreases. The x_1 - zero growth isoclines in turn, given by $\frac{dx_1}{dt} = 0$, are

$$x_1 = 0 \quad \text{... (6.28a)}$$

and $x_1 = k_1 - \alpha_{12}x_2$ ... (6.28b)

Below the line given by (6.28b), x_1 increases; above this, x_1 decreases (see Fig. 6.6)

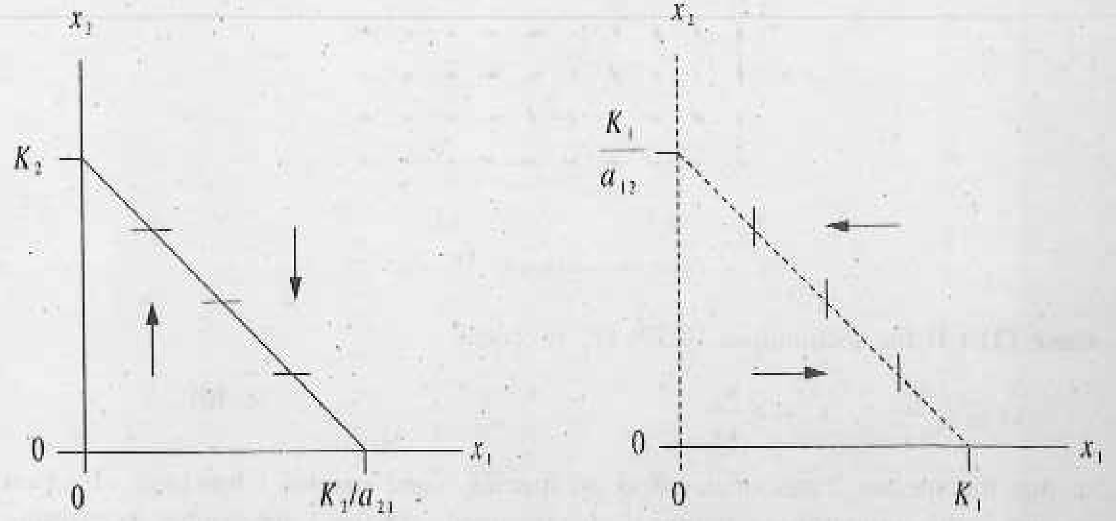


Fig. 6.6 : Zero-growth isoclines.

One of the isoclines (6.27b) and (6.28b) may be entirely above the other. Alternatively they may cross each other. There are four classes, depending on the relative position of x_1 and x_2 intercepts of these two zero growth isoclines. Each case corresponds to a different phase-portrait. Let us consider each one in turn.

Case (1) : If each intercept of the lines given by (6.27b) is greater than the corresponding intercept of that for (6.28b), so that

$$k_2 > \frac{k_1}{\alpha_{12}} \quad \text{or} \quad \alpha_{12} > \frac{k_1}{k_2}$$

$$\frac{k_2}{\alpha_{21}} > k_1 \quad \text{or} \quad \alpha_{21} < \frac{k_2}{k_1} \quad \text{... (6.29)}$$

x_2 excludes x_1 [see Fig. 6.7]. Thus, if species 2 has a relatively large effect on species 1 and species 1 has a relatively small effect on species 2, we expect that the species 1 will go extinct and the species 2 will approach its carrying capacity.

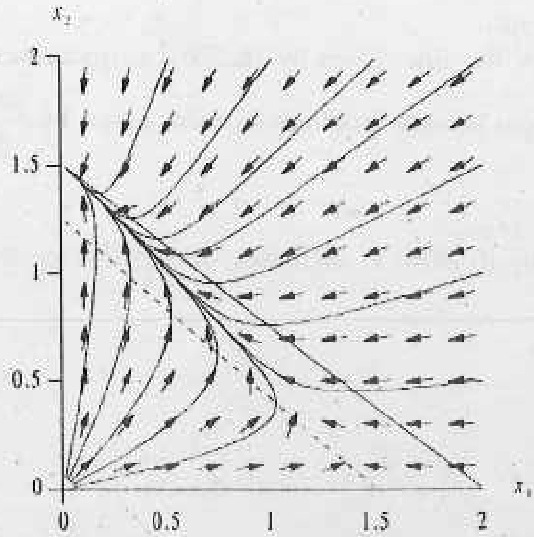


Fig. 6.7 Phase portrait for $a_{12} > \frac{K_1}{K_2}$, $a_{21} < \frac{K_2}{K_1}$

Case (2) : If the inequalities (6.29) are reversed

$$\alpha_{12} < \frac{k_1}{k_2} \quad , \quad \alpha_{21} > \frac{k_2}{k_1} \quad \ldots \quad (6.30)$$

so that the species 2 has small effect on species 1 and species 1 has large effect on species 2, the competitive outcome is also reversed : species 1 approaches its carrying capacity and species 2 goes to extinction (see Fig. 6.8).

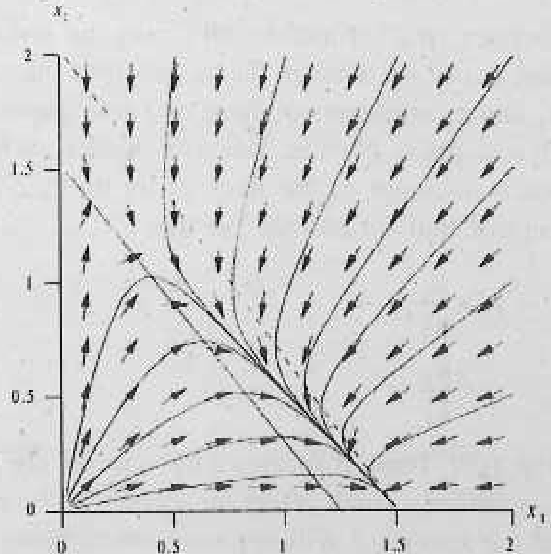


Fig. 6.8 Phase portrait for $a_{12} < \frac{K_1}{K_2}$, $a_{21} > \frac{K_2}{K_1}$

Case (3) : If $\alpha_{12} > \dfrac{k_1}{k_2}$, $\alpha_{21} > \dfrac{k_2}{k_1}$ ... (6.31)

In this case both the isoclines cross each other. In this case interspecific effects are large for both species. Two equilibria $(k_1, 0)$ and $(0, k_2)$ corresponding to the exclusions of one or the other species, are now both stable nodes (see Fig. 6.9). One or the other of the species will go extinct, depending on the initial conditions. There is a saddle point that lies between the two nodes.

Case (4) : If $\alpha_{12} < \dfrac{k_1}{k_2}$, $\alpha_{21} < \dfrac{k_2}{k_1}$ ... (6.32)

the equilibria $(k_1, 0)$ and $(0, k_2)$ are unstable saddle points and trajectories are drawn towards a stable node in the interior of the first quadrant. (Fig. 6.10).

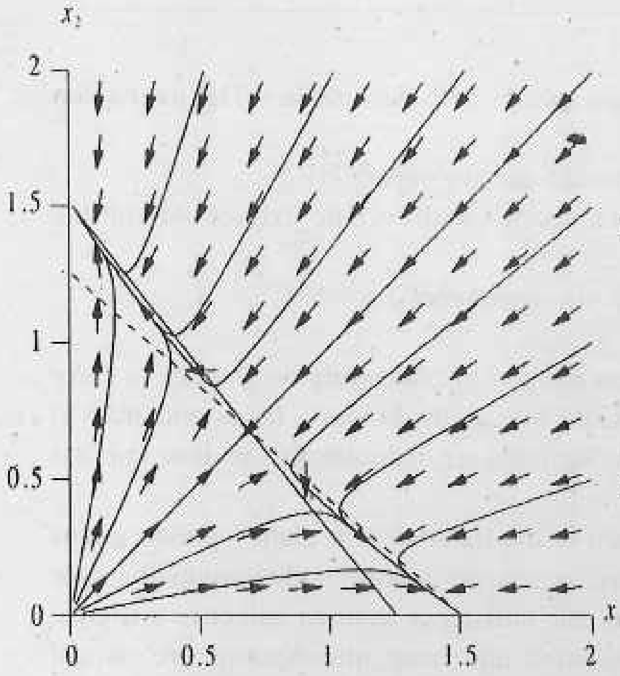


fig. (6.9) : $\alpha_{12} > \dfrac{k_1}{k_2}$

$\alpha_{21} > \dfrac{k_2}{k_1}$

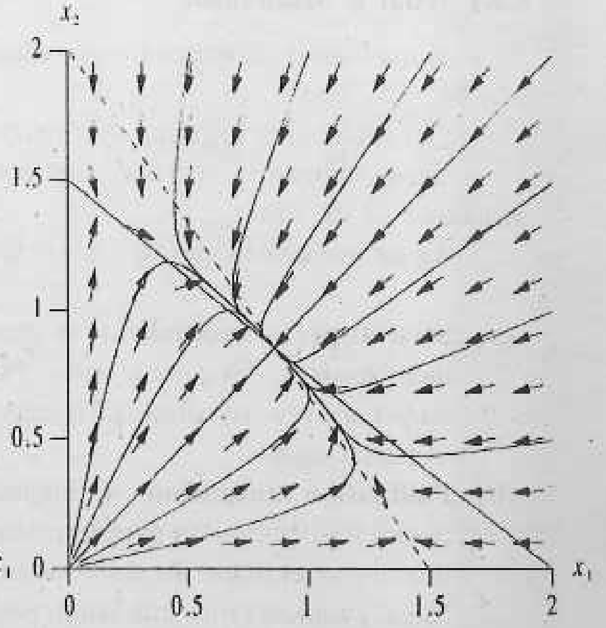


fig. (6.10) : $\alpha_{12} < \dfrac{k_1}{k_2}$

$\alpha_{21} < \dfrac{k_2}{k_1}$

Example (6.5) : For the competition model (6.26) show that there is no closed orbits (or periodic solution) within the first quadrant of (x_1, x_2) - plane.

Solution : Recall Bendixson - Dulac negative criterion. Let $B(x_1,x_2) = \dfrac{1}{x_1 x_2}$ Since the divergence

$$\frac{\partial}{\partial x_1}(Bx_1)+\frac{\partial}{\partial x_2}(Bx_2)=-\frac{r_1}{k_1}\frac{1}{x_1}-\frac{r_2}{k_2}\frac{1}{x_2} \quad \ldots \quad (6.23)$$

is strictly negative in the interior of the first quadrant, we can be sure that there are no closed orbits (or periodic solutions) contained entirely within the first quadrant.

For three out of the four cases we have considered, one species successfully excludes the other. Only in case (4), where interspecific effects were weak relative to intraspecific effects, did the two species coexist. This is the basis for Gause's "Principle of Competitive Exclusion". The experimental evidence is somewhat equivocal and there is considerable doubt about the universability of the principle.

6.4 Mutualism Models

6.4.1 What is Mutualism

Mutualism is an interaction in which species help one another. The interaction may be

(i) facultative meaning that two could survive separately

or (ii) obligatory, meaning that each species will become extinct without the assistance of the other.

Again mutualism can be classified into four types :

(a) **Seed-dispersal mutualism** : A great number of plants rely on animals to carry their seeds to favourable sites. Plants frequently produce fruits and nuts to attract and rewards dispersal agents. Squirells are undoubtedly the most familiar dispersal agents.

(b) **Pollination mutualism** : Pollination is the transfer of a plants' pollen grains before fertilization. In gymnosperms, the transfer is from pollen producing cone directly to an ovule. In angiosperms, the transfer is from an anther to a stigma. Most gymnosperms are wind pollinated and most angiosperms are animal pollinated. Angiosperm flowers often reward pollinators with nectar.

(c) **Digestive mutualism** : The guts of many animals are filled with bacteria, yeast and protozoa that help to breakdown food. Often, the host is unable to digest the food on its own. Cattle, deer and sheep rely on bacteria to breakdown plant cellulose and hemi cellulose into digestive sub-units.

(d) **Protection mutualism** : In 1874, the famous naturalist, Thomas Belt, described a remarkable mutualism - between ants and acacias. The genus Acacia contains a large number of trees and shrubs native to warm parts of both hemispheres.

Many of the plants in this genus house, support and employ ants. The ants guard the acacia against hervivores predators.

6.4.2 Lotka-Volterra Model of Mutualism :

Let us consider a simple model for a one-to-one facultative mutualism. This will be followed by obligatory mutualism. These two models are Lotka-Volterra competition models equation in which the negative competitive interactions has been turned into positive mutualistic interaction.

Let us consider a system of two species with population sizes (or densities) x_1 and x_2. Each species grows logistically in the absence of the other. Each species has per capita growth rate that decreases linearly with size (or density)

$$\frac{1}{x_1}\frac{dx_1}{dt} = r_1\left(1-\frac{x}{k}\right) \qquad \ldots \qquad (6.34a)$$

$$\frac{1}{x_2}\frac{dx_2}{dt} = r_2\left(1-\frac{x_2}{k_2}\right) \qquad \ldots \qquad (6.34b)$$

The introduction of mutualism between the species leads to the equations.

$$\frac{1}{x_1}\frac{dx_1}{dt} = r_1\left[1-\frac{(x_1-\alpha_{12}x_2)}{k_1}-\right] \qquad \ldots \qquad (6.35a)$$

$$\frac{1}{x_2}\frac{dx_2}{dt} = r_2\left[1-\frac{(x_2-\alpha_{21}x_1)}{k_2}-\right] \qquad \ldots \qquad (6.35b)$$

where the parameters α_{21} and α_{12} measure the strength of positive effect of species 2 on species 1 and of species 1 on species 2 respectively. The system of equations (6.35a) and (6.35b) can be written as

$$\frac{dx_1}{dt} = \frac{r_1}{k_1}x_1(k_1-x_1+\alpha_{12}x_2) \qquad \ldots \qquad (6.36a)$$

$$\frac{dx_2}{dt} = \frac{r_2}{k_2}x_2(k_2-x_2+\alpha_{21}x_1) \qquad \ldots \qquad (6.36b)$$

This is a model of facultative mutualism so far as

$$r_1>0,\ r_2>0,\ k_1>0,\ k_2>0 \qquad \ldots \qquad (6.37)$$

Each species can, in other words, survive without its mutualist.

Let us look at the zero-growth isoclines for this system. The x_2 - zero growth isoclines given by $\frac{dx_2}{dt}=0$ are

$$x_2 = 0 \quad \ldots \quad (6.38a)$$

$$\text{and} \quad x_2 = k_2 + \gamma_{21} x_1 \quad \ldots \quad (6.38b)$$

[see Fig. (6.11)].

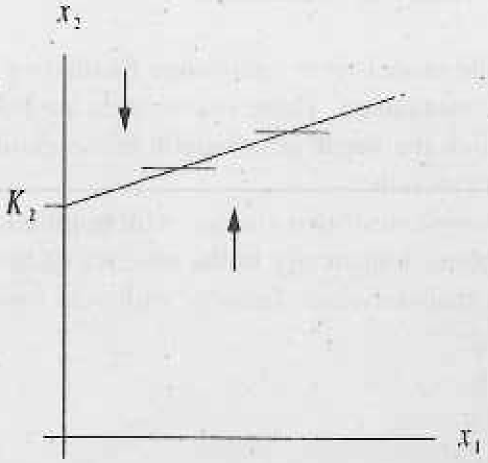


Fig. (6.11): x_2 - zero growth isoclines, $k_2 > 0$.

Below line (6.38b), x_2 increases; above this line, x_2 decreases.

The x_1 - zero growth isoclines given by $\frac{dx_1}{dt} = 0$, are

$$x_1 = 0 \quad \ldots \quad (6.39a)$$

$$\text{and} \quad x_1 = k_1 + \gamma_{12} x_2 \quad \ldots \quad (6.39b)$$

(see Fig. 6.12)

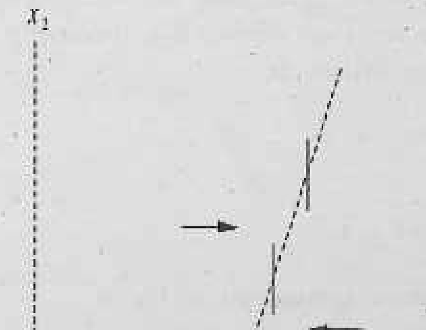


Fig. 6.12 : x_1 zero-growth isoclines; $k_1 > 0$.

To the left of line (6.39b), x_1 increases; to the right of this line, x_1 decreases. The zero-growth isoclines (6.38b) and (6.39b) may either converge or divergence. They converge if

$$\frac{1}{\alpha_{12}} > \alpha_{21}$$

$$\text{or } \alpha_{12}\alpha_{21} > 1 \qquad \ldots \qquad (6.40)$$

In this case, the two isoclines cross each other and orbits approach a stable node in the interior of the first quadrant (see Fig. 6.13). Since the slopes of the zero-growth isoclines are positive, the coordinates of this equilibrium are greater than the carrying capacity k_1 and k_2; each species surpasses its carrying capacity because of its mutualist.

$$\text{If } \alpha_{12}\alpha_{21} < 1 \qquad \ldots \qquad (6.41)$$

zero-growth isoclines (6.38b) and (6.39b) diverges. Now, the zero-growth isoclines do not cross and there is no non-trivial equilibrium in the first quadrant. The population undergo unlimited growth (see Fig. 6.14) in what has been called "an orgy of mutual benefaction".

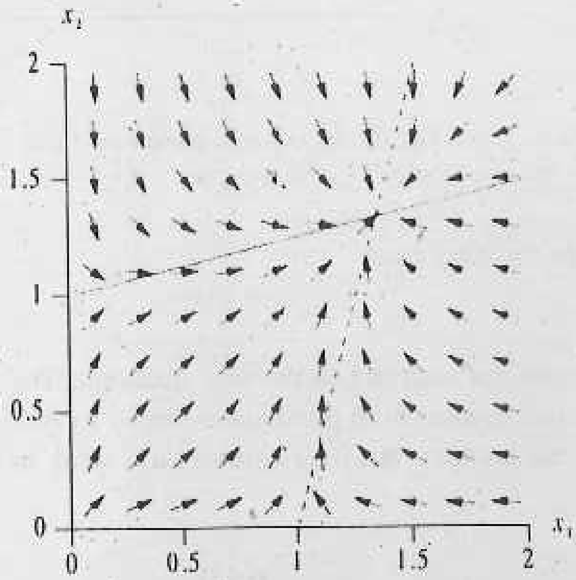


Fig. (6.13): Facultative mutuatism for $\alpha_{12}\alpha_{21} < 1$

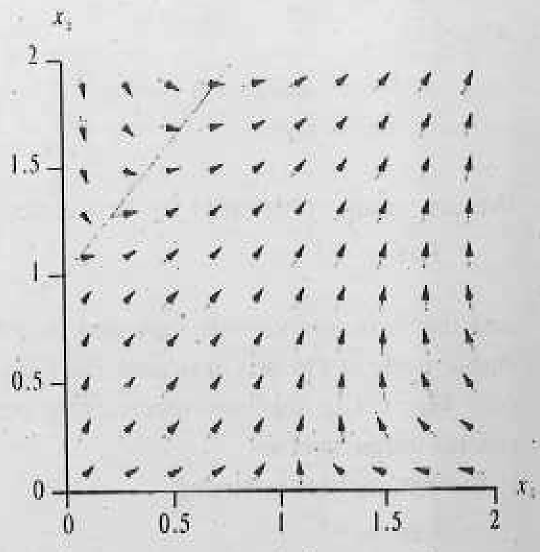


Fig. (6.14): Facultative mutuatism for $\alpha_{12}\alpha_{21} > 1$

Equations (6.36a) and (6.36b) can be used as a model for obligate mutulism if we assume that

$$r_1 < 0 \,,\; r_2 < 0 \,,\; k_1 < 0 \,,\; k_2 < 0 \qquad \ldots \qquad (6.42)$$

Neither species can now survive on its own; each species is banking on the other to save it. Equations (6.38) and (6.39) are still the correct equations for x_2 and x_1 zero-growth isoclines. However, since k_1 and k_2 are negative, the lines (6.38b) and (6.39b) look rather different [see Fig. 6.15 and 6.16]

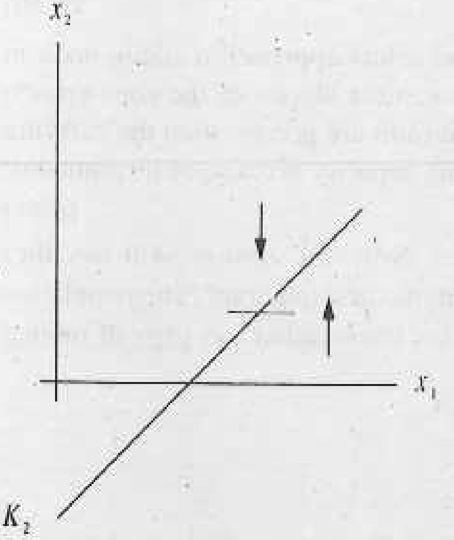


Fig. (6.15) : x_2 zero-growth isoclines. $k_2 < 0$

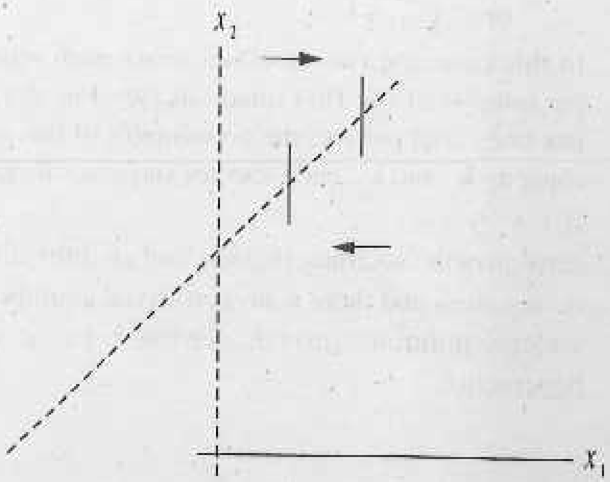


Fig. (6.16) : x_1 zero-growth isoclines. $k_1 < 0$

We are again confronted by two cases. In the first case

$$\alpha_{12}\beta_{21} < 1 \quad \dots \quad (6.43)$$

and the two isoclines (6.38b) and (6.39b) do not intersect in the first quadrant. The stable node at the origin is now the only equilibrium. Both populations decay to zero (see Fig. 6.17). the two species rely on one another, but interaction is too weak to rescue either species.

If the interaction is strong

$$\alpha_{12}\beta_{21} > 1 \quad \dots \quad (6.44)$$

the isoclines (6.38b) and (6.39b) do intersect in the first quadrant (see Fig. 6.18). There is now a saddle point in the first quadrant. If mutualist densities are low, both populations go extinct : the interaction is strong but there are too few mutualists to rescue either population. If mutualist densities are high, both species increase in another "orgy of mutual benefaction". Orbit now divergence to infinity

[see Fig.6.18]

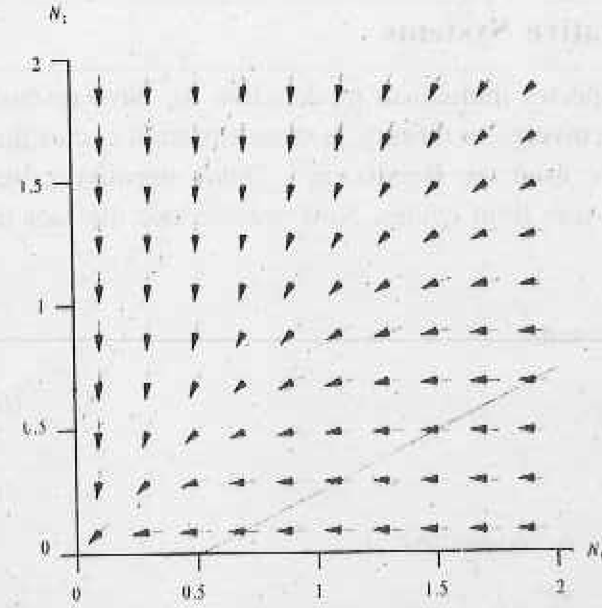


Fig. (6.17): Obligate mutualism phase phase-portrait for $\theta_{12}\theta_{21} < 1$

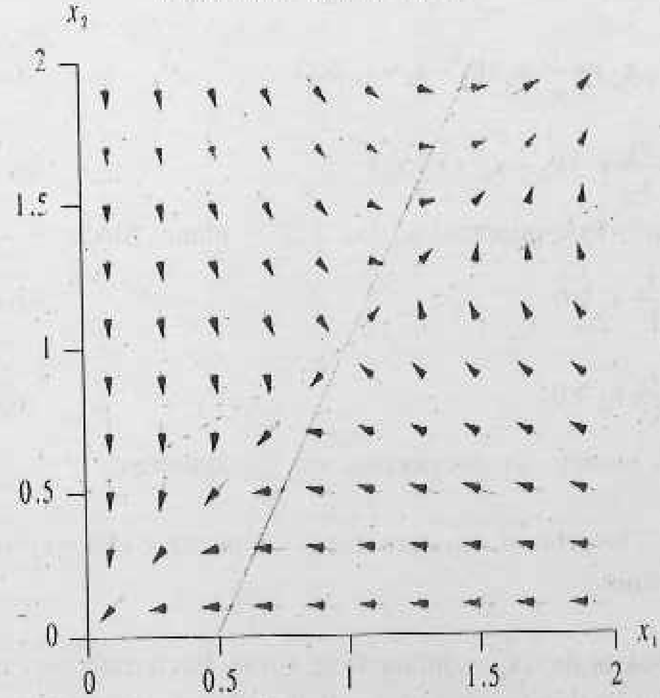


Fig. (6.18) Obligate mutualism phase phase-portrait for $\theta_{12}\theta_{21} > 1$

6.4.3 ❑ Cooperative Systems

All orbits of two-species mutualism models that we have discussed appear to tend to equilibrium or to diverge to infinity. Is there any limit cycles that we have missed? Previously we have used the Bendixson - Dulac negative criterion to prove that systems do not possess limit cycles. Now we can use the fact that our models are cooperative.

Definition : The system

$$\frac{dx_1}{dt} = f(x_1, x_2) \qquad \dots \qquad (6.45a)$$

$$\frac{dx_2}{dt} = g(x_1, x_2) \qquad \dots \qquad (6.45b)$$

defined on $D \subset R^2$, is cooperative if

$$\frac{\partial f}{\partial x_2} \geq 0 \quad , \quad \frac{\partial g}{\partial x_1} \geq 0 \qquad \dots \qquad (6.46)$$

for all $(x_1, x_2) \in D$

Example (6.6) :

Let $$f(x_1, x_2) = \frac{r_1}{k_1} x_1 (k_1 - x_1 + \alpha_{12} x_2) \qquad \dots \qquad (6.47a)$$

$$g(x_1, x_2) = \frac{r_2}{k_2} x_2 (k_2 - x_2 + \alpha_{21} x_1) \qquad \dots \qquad (6.47b)$$

on the (invariant*) first quadrant x_1 (xi, x_2) — plane. Since

$$\frac{\partial f}{\partial x_2} = \alpha_{12} \frac{r_1}{k_1} x_1 \geq 0 \qquad \dots \qquad (6.48a)$$

$$\frac{\partial g}{\partial x_1} = \alpha_{21} \frac{r_2}{k_2} x_2 \geq 0 \qquad \dots \qquad (6.48b)$$

those mutualism models are cooperative on this quadrant.

Theorem (6.3) : The orbit of a system that is cooperative either converge to equilibria or diverge to infinity.

Proof : Let us look at the $(\dot{x}_1, \dot{x}_2)$plane (Fig. 6.19). Each trajectory of a planer system generates an orbit in this $(\dot{x}_1, \dot{x}_2)$ plane. If the planer system that we are looking at is everywhere cooperative, the first quadrant of the $(\dot{x}_1, \dot{x}_2)$ plane is invariant*. To see this, we consider an orbit that attempts to leave the first quadrant by crossing the positive $\dot{x}_2$ - axis, In the light of the equation (6.45a) and our definition of cooperative system,

$$\frac{d}{dt}(\dot{x}_1)=\frac{d^2x_1}{dt^2}=\frac{\partial f}{\partial x_1}.\frac{dx_1}{dt}+\frac{\partial f}{\partial x_2}.\frac{dx_2}{dt} \quad \ldots \quad (6.49a)$$

$$=\frac{\partial f}{\partial x_2}.\frac{dx_2}{dt}>0 \text{ (since } \dot{x}_1=\frac{dx_1}{dt} \text{ is zero on } \dot{x}_2\text{-axis)} \quad \ldots \quad (6.49b)$$

on the positive $\dot{x}_2$ - axis. The orbit cannot cross the positive $\dot{x}_2$ - axis; for when the orbit crosses the positive $\dot{x}_2$ - axis, we must have $\dot{x}_1=\frac{dx_1}{dt}=0$ so that $\dot{x}_1$ can not. increase with time. By a similar argument the orbit can not cross the positive $\dot{x}_1$ - axis. Finally, the orbit cannot cross through the origin, since this would imply that the original trajectory passes through a rest point. Similar argument also shows that the third quadrant is also invariant*.

Ultimately (as $t\to\infty$) $\dot{x}_1$ and $\dot{x}_2$ are of constant sign. If we start in the first or third quadrant, we will stay there forever. If we start from the second or fourth quadrants, we may either stay in those quadrants or we may enter the first or third quadrants; which are invariants. Either way, x_1 and x_2 are ultimately monotonic function at time. This precludes limit cycles and implies that trajectoris either approach equilibria or diverge to infinity.

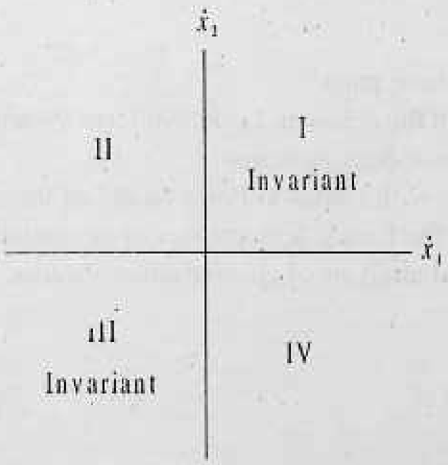


Fig. (6.19): ($\dot{x}_1$, $\dot{x}_2$) plane

* A set (a region) is invariant if an orbit starting from this set (or region) will stay there forever.

Exercises :

1. Determine the outcome of the competition the system

$$\frac{dx}{dt} = x(100 - 4x - y)$$

$$\frac{dx}{dt} = y(60 - x - y).$$

2. What is the outcome of a competition modelled by the system

$$\frac{dx}{dt} = x(2 - x - x^2 - y).$$

$$\frac{dy}{dt} = y(16 - 2x - x^2 - y).$$

3. For the mutualistic system

$$\frac{dx}{dt} = x(-20 - x + 2y)$$

$$\frac{dy}{dt} = y(-50 + x - 2y).$$

Find the equilibrium points and determine their stabilities.

6.5 ❑ Summary

This chapter consists of three parts' :

(i) The first one deals with the classical Lotka-Volterra system and the modification for a realistic model of predator-prey systems.

(ii) The second part deals with Lotka-Volterra model of the competition system,

(iii) The third deals with the Lotka-Volterra model of mutualism or symbiosis. It also includes the mathematical analysis of co-operative systems.

Unit ❑ 7 : Ecosystem Models

7.1 Introduction : Functional Groups

Objectives. The chapter consists of a brief account of dynamical modeling of ecosystems.

Structure:

7.1 Introduction: Functional groups
7.2 Linear Food-Chain: constant production
7.3 Logistic primary production
7.4 Material cycling: Linear Tropic Interaction
7.2 Summary

Two useful tools of studying a natural system are the laws of conservation of energy and mass. The study of ecosystem form the point of view of energy-flow (or energetics) was advocated by Odum and Odum (1975). The early ecosystem models, which used energy as the currency were not successful everywhere. Although energy inflows to many systems can be estimated quite accurately, the outflows are, however, hard to define and measure precisely. More recent works reveal that the energy flows inside an ecosystem occur in the form of chemically bound energy and are thus accompanied by flows of elemental nutrients. The inflows and outflows of nutrients are more easier to define and measure than their energetic counterpart. Modern ecosystem models thus adopt one or more essential elements, usually carbon, nitrogen or phosphorus, as their 'currency'. For the study of the flow of nutrient we focus from populations to functional groups - that is, groups of species which cause the passage of nutrient from one place to another. For example, in a model of grassland ecosystem we might skate over wealth of biological details and differentiate only between plants which are edible by harvivores and those which are not.

7.2 Linear Food-chain : Constant Production

(i) One Level system :

We consider an ecosystem with a single functional group, which we shall call 'plants'. We take carbon biomass as our currency and write the current carbon biomass density of plants as $P(t)$ gc/m^2. We assume that photosynthesis produces new biomass at a rate ϕ gc/m^2/day and that a plant of carbon mass u loses carbon through mortality and respiration at a rate $\delta_p u$ gc/day. The dynamics of this very simple system is described by a single equation

$$\frac{dP}{dt} = \phi - \delta_p P \qquad \cdots \qquad (7.1)$$

which has the steady-state,

$$P^* = \frac{\phi}{\delta_p} \qquad \cdots \qquad (7.2)$$

Thus the steady-state carbon density of plant (called steady-state standing stock) is given by the product of the primary production rate P and the average residence time of a carbon atom in a plant $\left(\frac{1}{\delta_p} \right)$. Let P = P* + p, p being the deviation from the steady-state value, then the equation (7.1) reduces to

$$\frac{dp}{dt} = -\delta_p p \qquad \cdots \qquad (7.3)$$

implying the stable steady-state for $\delta p > 0$.

(ii) Two-level System :

We now add a second functional group ('hervivores') to our ecosystem. Let H(t) gc/m^2 be the carbon biomass density of these organisms. We assume that respiration and mortality remove hervivore carbon at a per-capita rate δ_h/day, that hervivores feed exclusively on plants, with a linear functional response characterized by an attack rate α_h m^2/day/gc. This implies that in the presence of plant carbon density P, a hervivore of weight w consumes plant carbon at a rate $\alpha_h P w$ gc/day. The system dynamics is described by a pair of coupled differential equations

$$\frac{dP}{dt} = \phi - \delta_p P - \alpha_h PH \qquad \cdots \qquad (7.4)$$

$$\frac{dH}{dt} = \alpha_h PH - \delta_h H \qquad \cdots \qquad (7.5)$$

This system has two steady-states :

$$P^* = \frac{\delta_h}{\alpha_h}, \qquad H^* = \frac{1}{\delta_h}(\phi - \delta_p P^*) \qquad \cdots \qquad (7.6)$$

$$P^* = \frac{\phi}{\delta_p}, \qquad H^* = 0 \qquad \cdots \qquad (7.7)$$

Let us interprete the biological significance of the first-steady state (7.6). For a biologically possible or sensible solution, H* > 0. As a result we have,

$$\frac{\phi}{\delta_p} \geq \frac{\delta_h}{\alpha_h} \qquad \cdots \qquad (7.8)$$

which implies the decrease of steady-state standing stock of primary production with the presence of harvivores.

To study the local stability, we put

$$H = H^* + h \qquad P = P^* + p \qquad \ldots \qquad (7.9)$$

Putting these values in (7.4) and (7.5) we have the linearised equations,

$$\left.\begin{aligned} \frac{dh}{dt} &= \alpha_h H^* P \\ \frac{dp}{dt} &= -(\delta_p + \alpha_h H^*)\, p - \alpha_h P^* h \end{aligned}\right\} \qquad \ldots \qquad (7.10)$$

Seeking the solutions like $e^{\lambda t}$ shows that the eginevalues λ must satisfy the characteristic equation

$$\lambda^2 + (\delta_p + \alpha_h H^*)\lambda + \alpha_h^2 P^* H^* = 0 \qquad \ldots \qquad (7.11)$$

The constant term and the coefficient of λ are both unequivocally positive for biologically sensible (positive) steady-state. For the stability of the steady state (7.6), the eginevalues, that is, the roots of the characteristic equation (7.11) must have negative real parts.

The system of linear food-chains can be extended to a three-level system by addition of a functional group of 'consumers' which eat (only) hervivores.

7.3 Logistic Primary Production

The models discussed in section (7.2) are based on the assumption of constant primary production. Although these models are acceptable approximations for some systems, for many other the rate of primary production depends on the standing stock of primary producers. To investigate the implication of this, we shall modify our model of linear food-chain by assuming that in the absence of hervivory, the plant carbon biomass grow logistically to a carrying capacity k, that is,

$$\frac{dP}{dt} = rp\left(1 - \frac{P}{k}\right) \qquad \ldots . \qquad (7.12)$$

Two - level System :

We investigate a very simple system, in which the hervivores are the only additional tropic levels. We retain all other assumptions of section (7.2), so that the system dynamics are described by the equations,

$$\left.\begin{aligned}\frac{dP}{dt} &= r\,p\left(1-\frac{P}{k}\right)-\alpha_h PH \\ \frac{dH}{dt} &= \alpha_h PH - \delta_h H\end{aligned}\right\} \quad \cdots \quad (7.13)$$

This system is mathematically identical to the predator-prey model with logistic prey and linear predator functional response. It has two steady-states (K, 0) and (P*, H*) where

$$P^* = \frac{\delta_h}{\alpha_h}, \qquad H^* = \frac{r}{\alpha_h}\left(1-\frac{P^*}{K}\right) \quad \cdots \quad (7.14)$$

The co-existence steady-state (P*, H*) is biologically sensible, that is, $P^* > 0$ and $H^* > 0$, provided the required plant carbon biomass is less than the carrying capacity, that is,

$$K \geq \frac{\delta_h}{\alpha_h} \quad \cdots \quad (7.15)$$

Small deviations about steady-state (P*, H*) are described by

$$\left.\begin{aligned}\frac{dp}{dt} &= \frac{rP^*}{K}p - \alpha_h P^* h \\ \frac{dh}{dt} &= \alpha_h H^* p\end{aligned}\right\} \quad \cdots \quad (7.16)$$

and hence a characteristic equation,

$$\lambda^2 + \frac{rP^*}{K}\lambda + \alpha_h^2 P^* H^* = 0 \quad \cdots \quad (7.17)$$

Since both the constant terms and the coefficient of λ in this characteristic equation (7.17) are positive for all biologically sensible steady-states, we see that all such steady-states are necessarily stable. We can extend the model by addition of a consumer functional group to our logistic primary production model.

With a basic discussion of linear food chain and logistic primary production we close up the chapter on ecosystem. The models of food-chain and many other ecosystem problems are similar in both structure and dynamics to those which we have set out to describe interacting groups of unstructured population. We can, therefore, use our knowledge of such models to inform our view of the likely properties of ecosystem models.

7.4 ❑ Material Cycling : Linear Tropic Interaction :

In the previous sections we have considered the passage of elemental matter such as carbon, phosphorus or nitrogen, and resumed the primary production rate either to be a constant or a logistic function. The primary production is limited by some factors other than the elements being modelled : for example light. However, in closed system the elemental matter needed for primary production must be provided through recycling - by mortality and respiration in the case of carbon, or by mortality and excretion in the case of phosphorus and nitrogen. Few ecosystems outside the laboratory are closed to carbon, so in this section we explain the dynamic implications of closure of an elemental nutrient (i.e. nitrogen and phosphorus).

A Nutrient - Plant System :

The system model is very simple, it consists of a nutrient compartment, containing limiting nutrient at density N(t), and the plant functional group, which we now characterise by its limiting nutrient density P(t). We assume that the plants have a linear response, with slope ap and a mortality excretion rate δ_p. The system is closed, so any nutrient taken up by the plants is lost to the free nutrient pool, and all nutrient lost by plants due to death and excretion is immediately (or instantaneously) added to the nutrient pool. With these assumptions, the system dynamics are,

$$\left.\begin{aligned}\frac{dp}{dt} &= a_p NP - \delta_p P \\ \frac{dN}{dt} &= \delta_p P - a_p PN\end{aligned}\right\} \qquad \ldots \quad (7.18)$$

Equations (7.18) imply that

$$\frac{dN}{dt} + \frac{dP}{dt} = \frac{d}{dt}(P+N) = 0 \qquad \ldots \quad (7.19)$$

In other words, the total quantity of nutrient contained in the system is constant as it should be for a closed system. Representing the total amount of bound and unbound nutrient by S, the dynamical equations (6.18) reduce to

$$\frac{dP}{dt} = a_p NP - \delta_p P, \; N(t) + P(t) = S \qquad \ldots \quad (7.20)$$

Eliminating N, from (7.20), we have,

$$\frac{dP}{dt} = (a_p S - \delta_p)P\left[1 - \frac{a_p}{a_p S - \delta_p}P\right] \qquad \ldots \quad (7.21)$$

Writing $r_p = \alpha_p S - \delta_p$, $K_p = \frac{r_p}{\alpha_p}$, the equation (7.21) reduces to the form of logistic equation

$$\frac{dP}{dt} = r_p P\left[1 - \frac{P}{K_p}\right] \qquad \ldots \qquad (7.22)$$

From (7.22), we can conclude that the model has an unstable steady-state at $P = 0$ and a globally stable steady state at $P = K_p = (S - \delta_p / \alpha_p)$.

7.5 ❑ Summary

In the introduction (section 7.1) we have explained the concept of functional groups for the modeling of ecosystems. In sections(7.2) we have discussed the linear food-chain model of ecosystem with constant production. In section (7.3) we have modified the constant production model to the logistic primary production model. In section (7.4) we have discussed the basic concept of material cycling in functioning in ecosystems.

"For dealing with any natural phenomena-especially one of a vital nature, with all the complexity of living organism in type and habit—the mathematician has to simplify the conditions until they reach the attenuated character which lies in the power of this analysis"

—Kerl Pearson

Glossary of Ecological Terms

Abiotic : not biological or not relating to living organisms.

Abundance : large amount or large number of something.

Algae : tiny plants living in water or in moist condition.

Allele : one of two or more alternative forms of a gene; which can imitate each others form.

Allelopathy : harm caused by one plant to another plant, usually by producing a chemical substance.

Anther : part of a stamen which produces pollen.

Biomass : all living organisms in a given area or at a given tropic level expressed in terms of living or dry weight.

Biome : large ecological region characterized by similar vegetation and climate (such as desert, the tundra etc.).

Bion : single living organism in an ecosystem.

Biota : flora and fauna of a region.

Bloom : (a) flower; the blooms on the orchids have been ruined by frost (b) algae bloom = mass of algae which develop rapidly in a lake. 2. Verb, to flower. The plant blooms at night, some cacti only bloom once every seven years.

Carrying Capacity : maximum number of individuals of a species that can be supported in a given area.

Cellulose : carbohydrate which makes up a large percentage of plant matter.

Community : group of different organisms which live together in an area, and which are usually dependent on each other for existence.

Diversity : richness of the number of species in an area.

Ecology : study of relationship among organisms and the relationship between them and their physical environment.

Deep ecology : extreme form of ecological thinking where humans are considered as only one species among many in the environment.

Ecological balance (or balance of nature) : situation where relative number of organisms remain more or less constant.

Ecological succession : series of communities of organisms which follow on one after the other, until a climax community is established.

Ecospecies : subspecies of a plant.

Ecosphere : biosphere, part of the earth and its atmosphere where living organism exist.

Ecosystem : system which includes all organisms of an area and the environment in which they live.

Etholog : study of the behaviour of living organisms.

Evolution : heritage changes in organisms, which take place over along period involving many generation.

Genetics : study of the way the characteristics of an organism are inherited through genes.

Genome : all the genes in an individual.

Genotype : genetic composition of an organism.

Genus : group of closely-related species.

Green house : building made mostly of glass, used to raise and protect plants.

Green house effect : effect produced by accumulation of carbon dioxide crystals and water vapour in the upper atmosphere, which insulates the earth and raises the atmosphere temperature by preventing heat loss.

Habitat: type of environment in which an organism lives.

Heredity : occurrence of physical or mental characteristics in offspring which are inherited from their parents.

Immune : protected against an infection or allergic disease.

Niche : place in an ecosystem which a species is specially adapted to fit.

Ecological Niche : all the characters (chemical, physical and biological) that determine the position of an organism or species in an ecosystem, (commonly called the "role" or "profession" of an organism e.g. an aquatic predator, a terrestrial hervivore.

Omnivore : animal which eats any thing, both vegetation and meat.

Phenotype : physical characteristics of an organism which its genes produce, such as brown eye, height etc. compare genotype.

Pisci culture : fish farming; the breeding fish for food in special enclosures.

Terrestrial: referring to land; terrestrial animals : animals which live on dry land.

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